1999 VOL. X(XVIII)

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ON THE NORMALITY PROBLEM FOR THE FIRST ORDER DIFFERENTIAL OPERATORS

Abstract

A relation between the formal normality property of a minimal operator, generated by a first order differential expression with variable operator coefficients in a Hilbert space of vector-functions on a final segment, and operator coefficient of a differential expression that generates it, is studied in the paper.

All normal extensions of a minimal operator in terms of boundary conditions under admissible coefficients are also described.

A linear closed densely determined operator T in a Hilbert space H is called a formally normal one, if $D(T) \subset D(T^*)$ and $\|Tf\|_H = \|T^* f\|_H$ for any $f \in D(T)$. A formally normal operator is maximal, if it has no non-trivial formally normal extensions. Formally normal operator satisfying the condition $D(T) = D(T^*)$ (see [1]) is called a normal operator T.

A general theory of normal extensions of an unbounded formal normal operator in a Hilbert space is stated and developed in papers by E.A.Caddington (see [1]). This theory has not been investigated widely (for detail see [2]) in conformity to differential operators theory.

Denote by H a separable Hilbert space u, for convenience in indices, instead of $L_2(H,(0,1))$ - a Hilbert space of H - valued functions at the final segment $[0,1]$ - write H_1. Note that al encountered integrals are Lebesque ones.

Now consider a first order differential - operator expression of the form

$$l(u) = u'(t) + A(t)u(t), \quad 0 \le t \le 1,$$

where $A(t)$ at each fixed $t \in [0,1]$ is a linear bounded operator in H, and the operator-functions $A(t)$ and $A^*(t)$ are strongly continuous on $[0,1]$.

A formally adjoint differential-operator expression $l(\cdot)$ in $L_2(H,(0,1))$ has the form

$$l^+(v) = -v'(t) + A^*(t)v(t), \quad 0 \le t \le 1.$$

Denote by $L_0(L_0^+)$ minimal and by $L(L^+)$ maximal operators generated by the expression $l(\cdot)$ by adjoint expression $l^+(\cdot)$ in $L_2(H,(0,1))$ (see [3]).

Thus, the inclusions

$$L_0 \subset L, \quad L_0^+ \subset L^+$$

hold.

In this paper, a relation between the property of a formal normality of a minimal operator L_0 and operator coefficients $A(t)$ are studied, and all normal extensions of a minimal operator in terms of boundary conditions under admissible coefficients are described.

1. First prove the following

Lemma 1.1. *Let* $a(t)\in C[0,1]$. *If for each function* $\varphi(t)$ *from* $\overset{0}{W}{}_2^1(0,1)$, *the condition*

$$\int_0^1 a(t)\varphi^2(t)dt = 0 \tag{1.1}$$

is fulfilled, then $a(t)=0, 0<t<1$.

Proof. If $a(t)\geq 0$ (≤ 0), $0\leq t\leq 1$, it follows from (1.1) that $a(t)=0$, $0<t<1$. Now assume that $a(t)$ is non-negative in the intervals $\Delta_n=(\alpha_n,\beta_n)\subset[0,1]$, $n=1,2,...$, and in remaining places, $[0,1]\setminus\bigcup_{(n)}\Delta_n$ it is non-positive. Assume that $a(t)$ is non-negative only in one interval (α,β):

$$a(t)\geq 0, \quad t\in(\alpha,\beta),$$
$$a(t)\leq 0, \quad t\in[0,1]\setminus(\alpha,\beta).$$

Consider a function of the following form:

$$\varphi_{\alpha\beta}(t)=\begin{cases}0, & 0\leq t\leq\alpha\\ (t-\alpha)(t-\beta), & \alpha\leq t\leq\beta,\\ 0, & \beta\leq t\leq 1.\end{cases}$$

$\varphi_{\alpha\beta}(t)\in\overset{0}{W}{}_2^1(0,1)$ and from the relation (1.1) we get

$$\int_0^1 a(t)\varphi_{\alpha\beta}^2(t)dt=\int_\alpha^\beta a(t)\varphi_{\alpha\beta}^2(t)dt=0.$$

Considering $a(t)\varphi_{\alpha\beta}^2(t)\in C[\alpha,\beta]$, $a(t)\varphi_{\alpha\beta}^2(t)\geq 0$, $\varphi_{\alpha\beta}(t)\neq 0$, $t\in(\alpha,\beta)$, we have $a(t)=0$, $t\in(\alpha,\beta)$.

If now in (1.1) instead of the function $\varphi(t)$ we take by turns the functions

$$\varphi_{0\alpha}(t)=\begin{cases}t(t-\alpha) & 0\leq t\leq\alpha\\ 0, & \alpha\leq t\leq 1,\end{cases}\qquad \varphi_{\beta 1}(t)=\begin{cases}0, & 0\leq t\leq\beta,\\ (t-\beta)(t-1), & \beta\leq t\leq 1\end{cases}$$

we find

$$a(t)=0, \quad t\in(0,\alpha) \text{ and } a(t)=0, \quad t\in(\beta,1).$$

So, $a(t)=0$, $t\in(0,\alpha)\cup(\alpha,\beta)\cup(\beta,1)$. Since $a(t)$ is continuous on $[0,1]$, then $a(t)=0$, $t\in(0,1)$.

A general situation related with the sign of the function $a(t)$ is proved similarly. It is valid the following

Theorem 1.2. *Let the operator-functions* $A(t)$ *and* $A^*(t)$ *are strongly continuous, and* $A_R(t)$ *is strong continuously differentiable on* $[0,1]$ *in* H. *In order the minimal operator* L_0 *be formally normal in* $L_2(H(0,1))$, *it is necessary and sufficient to fulfill the condition*

$$A^*(t)A(t)-A(t)A^*(t)=2A_R'(t), \quad 0<t<1. \tag{1.2}$$

Proof. Necessity. Let L_0 be formally normal in $L_2(H,(0,1))$. Then, it is valid the relation

$$\|L_0u\|_{H_1}=\|L^+u\|_{H_1}, \quad u(t)\in D(L_0), \text{ i.e.}$$

$$2(u',A_R(t)u)_{H_1}+2(A_R(t)u,u')_{H_1}+(A^*(t)A(t)-A(t)A^*(t)u,u)_{H_1}=0,$$

and this relation brings to

$$2[(u(1),A_R(1)u(1))_H-(u(0),A_R(0)u(0))_H]+ \\ ((A^*(t)A(t)-A(t)A^*(t)-2A_R'(t))u,u)_{H_1}=0 \qquad (1.3)$$

If we take $u(t)=\varphi(t)f,\quad \varphi(t)=\overline{\varphi}(t)\in \overset{0}{W}{}_2^1(0,1),\quad f\in H$, then $u(t)\in D(L_0)$ and we have

$$\int_0^1((A^*(t)A(t)-A(t)A^*(t)-2A_R'(t))f,f)_H\varphi^2(t)dt=0.$$

A realvalued function

$$a_f(t)=((A^*(t)A(t)-A(t)A^*(t)-2A_R'(t))f,f)_H= \\ =\|A(t)f\|_H^2-\|A^*(t)f\|_H^2-2(A_R'(t)f,f)_H,\quad f\in H,$$

is continuous on $[0,1]$. Then, by applying Lemma 1.1, from the last relation we get

$$a_f(t)=0,\quad 0<t<1,\quad f\in H.$$

Hence, we can conclude that

$$A^*(t)A(t)-A(t)A^*(t)=2A_R'(t),\quad 0<t<1.$$

Sufficiency. *In scopes of restrictions of the theorem, prove that L_0 is formally normal. Let $u(t)\in D(L_0)$. Since*

$$L^+u(t)=-(u'+A(t)u)+2A_R(t)u,\quad u'+A(t)u\in L_2(H,(0,1)),$$

$$\int_0^1\|A_R(t)u\|_H^2dt\le\int_0^1\|A_R(t)\|_H^2\|u\|_H^2dt\le\max_{[0,1]}\|A_R(t)\|_H^2\int_0^1\|u(t)\|_H^2dt<+\infty,$$

then $L^+u(t)\in L_2(H,(0,1))$, i.e. $D(L_0)\subset D(L^+)$.

This completes the proof of the theorem.

Corollary 1.3. *Let $A(t)$ at each t from $[0,1]$ be bounded, normal in H, and $A_R(t)$ be strongly continuously differentiable on $[0,1]$ in H. In order L_0 to be formally normal, it is necessary and sufficient the fulfilment of the condition*

$$A_R(t)=const,\quad 0<t<1.$$

Corollary 1.4. *If the operator-functions $A(t)$ and $A^*(t)$ are strongly continuously differentiable on $[0,1]$ in H, and L_0 is formally normal in $L_2(H,(0,1))$ the relations*

$$A_j(t)A_R(t)-A_R(t)A_j(t)=iA_R'(t),\quad 0<t<1,$$

$$(A_R'(t)f,f)_H=2Jm(A_R(t)f,A_j(t)f)_H,\quad 0<t<1,\quad f\in H,$$

$$|(A_R'(t)f,f)_H|\le 2|(A_R(t)f,A_j(t)f)_H|,\quad 0<t<1,\quad f\in H,$$

$$(A_R'(t)f,f)_H=\frac{1}{2}\left\{\|A_R(t)f\|_H^2-\|A^*(t)f\|_H^2\right\},\quad 0<t<1,\quad f\in H$$

are valid.

Corollary 1.5. *If $A_R(t)=0,\quad 0\le t\le 1$, then (1.2) is fulfilled automatically, and a minimal operator is anti-symmetric in $L_2(H,(0,1))$.*

2. In this item, we describe normal extensions of the minimal operator L_0.

Theorem 2.1. *Let $A_R(t)$ be strongly continuously differentiable on $[0,1]$ in H, and $A^*(t)A(t)-A(t)A^*(t)=2A_R'(t)$, $0<t<1$.*

Each normal extension $\widetilde{L}$ of the minimal operator L_0 is generated by differential expression $l(u)$ and boundary condition

$$u(1)=Wu(0), \tag{2.1}$$

where W is a unitary operator in H, and $WA_R(0)=A_R(1)W$.

A unitary operator W is uniquely determined by the extension $\widetilde{L}$, i.e. $\widetilde{L}=L(W)$.

On-the contrary, the contraction of the maximal operator L in the set of vector-functions $u(t)\in D(L)$ satisfying the condition (2.1) with the unitary operator W of peculiarity $WA_R(0)=A_R(1)W$, represent a normal extension of the minimal extension of the minimal operator L_0 in $L_2(H,(0,1))$.

Proof. Assume that $\widetilde{L}$ is a normal extension of the minimal operator L_0 in $L_2(H,(0,1))$. Then, the operator

$$\widetilde{L}_J=\left(\widetilde{L}-\widetilde{L}^*\right)/(2i)$$

is a self-adjoint extension of the closed, symmetric minimal operator L_0^J, generated by the formally symmetric differential expression $l^J(u)=\left(l(u)-l^*(u)\right)/(2i)=-iu'+A_J(t)u$. In this case, we can prove that the triple

$$(\mathcal{H},\gamma_1,\gamma_2),\ \mathcal{H}=H,\quad \gamma_1(u)=(u(o)-u(1))/\sqrt{2},\quad \gamma_2(u)=(u(o)-u(1))/\left(\sqrt{2}i\right),$$

is the space of boundary values of the minimal operator L_0^J (see [2], [4]). Then, in H, there exists a unique unitary operator W such that, it is fulfilled the relation

$$(W-E)\gamma_1(u)+i(W+E)\gamma_2(u)=0,\quad u(t)\in D\left(\widetilde{L}_J\right),$$

here E is an identity operator in H. The last condition is equivalent to the condition

$$u(1)=Wu(0).$$

Operator $\widetilde{L}_R=\left(\widetilde{L}+\widetilde{L}^*\right)/2$ is the self-adjoint extension of the minimal operator L_0^R, generated by the formally symmetric expression

$$l^R(u)=A_R(t)u(t)\ \text{ in } L_2(H,(0,1)):$$

$$\widetilde{L}_Ru=A_R(t)u(t),\quad u(t)\in D\left(\widetilde{L}_R\right)=D\left(\widetilde{L}\right).$$

In the given case from the second condition of the normality it follows

$$2\left[(u(1),A_R(1)u(1))_H-(u(0),A_R(0)u(0))_H\right]+\left(\left(A^*(t)A(t)-A(t)A^*(t)-2A_R'(t)\right)u,u\right)_{H_1}=0.$$

The second summand equals to zero, because $\widetilde{L}$ is formally normal. Thus

$$(u(1),A_R(1)u(0))_H-(u(0),A_R(0)u(0))_H=0,\quad u(t)\in D\left(\widetilde{L}\right).$$

Since $A_R(1)$ and $A_R(0)$ are bounded operators in H, then there exists the number $\gamma\in R^1$ such that

$$A_R(0)\geq\gamma+1,\quad A_R(1)\geq\gamma+1.$$

If in the following equality

$$(u(1),(A_R(1)-\gamma)u(1))_H-(u(0),(A_R(0)-\gamma)u(0))_H+\gamma\|u(1)\|^2-\gamma\|u(0)\|^2=0$$

we take into account the condition (2.1), we get

$$\left\|(A_R(1)-\gamma)^{1/2}u(1)\right\|_H=\left\|(A_R(0)-\gamma)^{1/2}u(0)\right\|_H.$$

Then, there exists an isometric operator V in H such that

$$\left(A_R^{(1)}-\gamma\right)^{1/2}u(1)=V(A_R(0)-\gamma)^{1/2}u(0),$$

i.e.

$$u(1)=(A_R(1)-\gamma)^{-1/2}V(A_R(0)-\gamma)^{1/2}u(0),\quad u(t)\in D(\widetilde{L}).$$

It becomes clear from abovesaid that a self-adjoint extension $\widetilde{L}_J$ determined by two boundary conditions

$$(W-E)\gamma_1(u)+i(W+E)\gamma_2(u)=0,$$
$$(W_1-E)\gamma_1(u)+i(W_1+E)\gamma_2(u)=0,$$
$$W_1=(A_R(1)-\gamma)^{-1/2}V(A_R(0)-\gamma)^{1/2},\quad u(t)\in D\left(\widetilde{L}_J\right)$$

Since each self-adjoint extension is determined only by a unitary operator W (see [4]), then $W=W_1$, i.e.

$$W=(A_R(1)-\gamma)^{-1/2}V(A_R(0)-\gamma)^{1/2}.$$

Hence we have

$$V=(A_R(1)-\gamma)^{1/2}W(A_R(0)-\gamma)^{-1/2}.$$

It follows from the relation $V^*V=E$, that

$$W^*A_R(1)u=A_R(0),\quad \text{i.e.}\quad WA_R(0)=A_R(1)W.$$

Now, on the contrary, assume that W is a unitary operator in H with property $WA_R(0)=A_R(1)W$. Denote by $L(W)$ a contraction of the maximal operator L, in a set of vector-functions $u(t)$, satisfying the condition (2.1). It is clear, that $L_0\subset L(W)\subset L$. Prove that $L(W)$ is normal in $L_2(H,(0,1))$.

A self-adjoint extension $L^*(W)$ is generated by a formally adjoint differential expression

$$l^+(v)=-v'(t)+A^*(t)v(t)$$

and boundary condition

$$V(0)=W^*V(1) \tag{2.2}$$

in the space $l_2(H,(0,1))$. The boundary conditions (2.1) and (2.2) determine the same set of vector-functions in $L_2(H,(0,1))$. On the other hand, for each $u(t)\in D(L(W))$ we have

$$\|L(W)u\|_{H_1}^2-\|L^*(W)u\|_{H_1}^2=2\left(u(0),\left(W^*A_R(1)W-A_R(0)\right)u(0)\right)_H+$$
$$+\left(\left(A^*(t)A(t)-A(t)A^*(t)-2A_R'(t)\right)u,u\right)_{H_1}=0$$

If at the latter we consider the conditions of the theorem, we get

$$\|L(W)u\|_{H_1}=\|L^*(W)u\|_{H_1},\quad u(t)\in D(L(W))=D(L^*(W)).$$

This completes the proof of the theorem.

Example. *Let* $\dim H=1$. In the space $L_2(0,1)$ we consider the differential expression

$$l(u)=u'+e^{it}u.$$

We easily see that

$$A^*(t)A(t) - A(t)A^*(t) - 2A'_R(t) = 2\sin t \neq 0, \quad 0 < t < 1.$$

Then by Theorem 1.2 a minimal operator is not formally normal.

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Received September 2, 1999; Revised October 1, 1999.
Translated by Aliyeva E.T.