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DECOMPOSITIONAL DESCRIPTION OF BESOV'S UMD-VALUED DOMINANT SPACE $S^{\alpha}_{p,\theta}B(R^n,E)$

Abstract

Dominant mixed derivative Besov spaces are considered. The values of the functions from these spaces are defined by the necessary and sufficient conditions for the decompositional description of entire functions from E-Banach spaces.

A number of variational and boundary- value problems lead to S-spaces of functions with a dominant mixed derivative. S.M. Nikolsky [1] first systematically studied S-spaces. He created a closed theory of these spaces, constructed on the basis of W and H- spaces. T.A. Amanov [2] and A.D. Jabrailov [3] independently introduced and studied the properties of S-spaces for number functions, constructed on the basis of O.B. Besov's B- spaces that are Banach spaces (see also [4]).

In this paper, Besov's Banach- valued dominant spaces $S^{r}_{p,\theta}B(R^n,E)$ are determined, and a decompositional description of these spaces is given. Decompositional description for Banach-valued Besov functions $B^{r}_{p,\theta}(R^n,E)$ was obtained by V.S. Guliev [7]. Similar problems for Sobolev's Banach- valued spaces $W^{l}_{p}(R^n,E_0,E)$ were studied by V.B. Shahmurov [5], and V.S. Guliev [6]. They were studied by M.S. Jabrailov [8] for Besov's Banach- valued spaces $B^{r}_{p,\theta}(R^n,E_0,E)$.

Let N be a set of positive integers, $N_0 = N\cup\{0\}, n\in N, R^n = (-\infty,+\infty)^n$ be n-dimensional Euclidean space, e^i is of ort standard basis in R^n, $x=(x_1,\dots,x_n)=\sum_1^n x_i e_i$, $y\in R^n$, $xy=\sum_1^n x_i y_i$, $|x|=(x\cdot x)^{1/2}$.

We denote by $\alpha=(\alpha_1,\dots,\alpha_n)$, $k=(k_1,\dots,k_n)$ a multiindex with integral non-negative components $|\alpha|=\sum_1^n \alpha_i$, $(\alpha,k)=\sum_1^n \alpha_i k_i$. Assume $D_j=\dfrac{\partial}{\partial x_j}$, $D^k = D_1^{k_1}\dots D_n^{k_n}$.

Let E be a Banach space, $p=(p_1,\dots p_n)$ be a vector with components satisfying the inequalities $1\le p_i\le\infty, (i=1,\dots,n)$.

We denote by $L_p(R^n,E)$ a space of E- valued functions $f(x)$ and $x\in R^n$ strongly measurable on R^n, and for which the norm

$$\|f\|_{L_p(R^n,E)} = \|f\|_{p,R^n,E} = \|f\|_{(p_1,\dots,p_n),R^n,E} =$$

$$=\left\{\int_R\left[\cdots\left\{\int_R\left(\int_R\|f(x)\|_E^{p_1}dx_1\right)^{p_2/p_1}dx_2\right\}^{p_3/p_2}\cdots\right]^{p_n/p_{n-1}}dx_n\right\}^{1/p_n}$$

is finite.

Note that if $p=(p,\dots,p)$,

$$\|f\|_{L_p(R^n,E)}=\|f\|_{L_p(R^n,E)},$$

Let us agree to write later $p\ge q$ or $p>q$, where $p=(p_1,\dots,p_n)$, $q==(q_1,\dots,q_n)$, if correspondingly $p_i\ge q_i$ $(i=1,\dots,n)$ or $p_i>q_i$ $(i=1,\dots,n)$; in particular, $1\le p\le\infty(1=(1,\dots,1),\infty=(\infty,\dots,\infty))$ means that $1\le p_i\le\infty$ $(i=1,\dots,n)$.

Denote by the symbol e_n a set of indices $1,\dots,n$, its arbitrary subset denote by e (the empty subset $\emptyset$ is contained among these subsets).

If n - dimensional vector $r=(r_1,\dots,r_n)$ is given with non-negative components, r^e denotes the vector $(r_1^e,\dots,r_n^e)$ with components $r_j^e=r_j$, $j\in e$, $r_j^e=0$, $j\notin e$ (in such a way the null-vector $r^\emptyset=(0,\dots,0)$ corresponds to the empty subset $\emptyset$).

For E - valued function $f(x)$, given on R^n, define a first order difference in the direction of j-th axis

$$\Delta_{h,j}=f(x_1,\dots,x_{j-1},x_j+h,x_{j+1},\dots,x_n)-f(x)$$

and generally of s-th order

$$\Delta_{h,j}^s=\Delta_{h,j}\dots\Delta_{h,j}f(x)$$

at the point x with the step h. If $k=(k_1,\dots,k_2)$ is an entire non-negative vector, i.e. $k_j\ge 0$, $j=1,\dots,n$, a mixed difference of the order k with a vector step $h=(h_1,\dots,h_n)$ is determined by the equality

$$\Delta_h^k=\Delta_{h_1,1}^{k_1}\dots\Delta_{h_n,n}^{k_n}f(x).$$

Definition 2. *Let* $k=(k_1,\dots,k_2)\in N^n$, $r=(r_1,\dots,r_n)\in N_0^n$, $k_i>r_i\ge 0$, $i=1,\dots,n$, $p=(p_1,\dots p_n)\in[1,\infty]^n$, $\theta\in[1,\infty]$. *A linear normed space of* E *- valued functions* f, *defined on* R^n *with a finite norm*

$$\|f\|_{S_{p,\theta}^rB(R^n,E)}=\|f\|_{p,E}+\|f\|_{s_{p,\theta}^rb(R^n,E)}=\|f\|_{p,E}+$$

$$+\sum_{e\subset e_n}\left\{\int_0^\infty\cdots\int_0^\infty\left\|\Delta_{h^e}^{k^e}f(\cdot)\right\|_{p,E}^\theta\prod_{j\in e}\frac{dh_j}{h_j^{1+r_j\theta}}\right\}^{1/\theta},1\le\theta\le\infty \tag{1}$$

we call the space $S_{p,\theta}^rB(R^n,E)$.

Respectively, the space $S_{p,\theta}^rB(R^n,E)=S_p^rH(R^n,E)$ is determined by the finiteness requirement of the norm

$$\|f\|_{S_p^rH(R^n,E)}=\|f\|_{p,E}+\|f\|_{s_p^rh(R^n,E)}=\|f\|_{p,E}+$$

$$+\sum_{e\subset e_n}\sup_{h>0}\left\{\left\|\Delta_{h^e}^{k^e}f\right\|_{p,E}\prod_{j\in e}h_j^{-r_j}\right\}. \tag{2}$$

Say that a Banach space E is ζ convex (or convex by Burkholder) (see for instance [9]), if there exists a symmetric function $\zeta(a,b)$ on $E\times E$, that is convex on each of variables satisfying the conditions

$$\zeta(0,0)>0,\ \ \zeta(a,b)\le\|a+b\|_E \quad \text{for} \quad \|a\|_E=\|b\|_E=1.$$

Describe the space $S'^{r}_{p,\theta}B(R^n,E)$ in terms of Fourier transformation of functions belonging to them. It seems especially simple in case when E is convex by Burkholder, i.e. $E\in UMD$ (see, for instance [6], [7], [9]). This is the case we are interested in.

Let $S=S(R^n)$ be a Schwartz space of principal functions, $S\otimes E$ is a linear span of the functions $(\varphi\otimes b)(x)=\varphi(x)b$, $\varphi\in S$, $b\in E$. Totality of linear continuous mappings of $S(R^n)$ in E is called the space $S'(E)=S'(R^n,E)$ of E-valued generalized functions on R^n.

Fourier transformation $\hat{\varphi}$ and the inverse Fourier transformation $\hat{\varphi}$ for $\varphi\in S$ are defined by formulas

$$\hat{\varphi}(\lambda)=(F\varphi)(\lambda)=(2\pi)^{-n/2}\int_{R^n}\varphi(x)e^{-i\lambda x}dx,$$

$$\hat{\varphi}(\lambda)=(F^{-1}\varphi)(x)=(2\pi)^{-n/2}\int_{R^n}\varphi(\lambda)e^{i\lambda x}d\lambda.$$

Fourier transformation and the inverse Fourier transformation of E- valued generalized functions f are defined by the formulas $(\hat{f},\varphi)=(f,\hat{\varphi})$, $(\check{f},\varphi)=(f,\check{\varphi})$, $\varphi\in S$.

If $f\in L_1(R^n,E)$, we say that $\lambda\in R^n$

$$\hat{f}(\lambda)=(Ff)(\lambda)=(2\pi)^{-n/2}\int_{R^n}f(x)e^{-i\lambda x}dx,$$

where an integral is in Bohner's sense.

Let Z^n denote a totality of integer vectors $m=(m_1,\ldots,m_n)$, $a=(a_1,\ldots,a_n)>1$, $a^m=a_1^{m_1}\cdots a_n^{m_n}$, $[a^m]=(a_1^{m_1},\ldots,a_n^{m_n})$, $[m,a]=(m_1a_1,\ldots,m_na_n)$.

Assume $\Pi_m(\alpha)=\{\lambda;\lambda\in R^n, a_j^{m_j-1}\le|\lambda_j|<a_j^{m_j}, j=1,\ldots,n\}$.

Denote by χ_A a characteristic function of the set A. The function

$$f_m(x,a)=F^{-1}\left(\chi_{\Pi_m(\alpha)}\hat{f}\right)$$

is correctly defined and belongs to $L_p(E)$ for the function $f\in L_p(E)$, $1<p<\infty$.

It is clear that the sets $\Pi_m(\alpha)$ do not intersect and they form the partition $T=\{\Pi_m(\alpha)\}$ of the space $(R^1\setminus\{0\})^n$

$$(R^1\setminus\{0\})^n=\bigcup_m\Pi_m(\alpha),$$

where the join is taken on all $m\in Z^n$. The representation of the function $f\in L_p(E)$ in the form of

$$f(x)\sim\sum_m f_m(x,a)$$

is called the decomposition of f in bundles.

Theorem 1. *In order that the function* f *belong to the space* $S^{\alpha}_{p,\theta}B(R^n,E)$, $\alpha=(\alpha_1,\dots,\alpha_n)>0$, $1\le\theta\le\infty$, $1\le p\le\infty$ *it is necessary and sufficient to be presented converging in metrics* $L_p(E)$ *by series*

$$f(x)=\sum_{m\ge 0}Q_{[\alpha^m]}(x) \tag{3}$$

($Q_{[\alpha^m]}$ *is an entire* E*- valued function of degree* $[\alpha^m]$*) for which the finite value* $\left(a^{[m,\alpha]\theta}=a_1^{m_1\alpha_1\theta}\cdots a_n^{m_n\alpha_n\theta}\right)$

$$\left\{\sum_{m\ge 0}a^{[m,\alpha]\theta}\left\|Q_{[\alpha^m]}\right\|^{\theta}_{p,E}\right\}^{1/\theta}<\infty. \tag{4}$$

And what is more, the lower bound of this value on all possible expansions (4) is equivalent to the norm $\|f\|_{S^{\alpha}_{p,\theta}B(R^n,E)}$.

Here we can show the sequence $T=\{T_m\}$ *of linear operators* T_m*, associating the function* f *for functions* $Q_{[\alpha^m]}\equiv Q^T_{[\alpha^m]}$ *such that*

$$\|f\|_{S^{\alpha}_{p,\theta}B(R^n,E)}\cong\left\{\sum_{m\ge 0}a^{[m,\alpha]\theta}\left\|Q^T_{[\alpha^m]}\right\|^{\theta}_{p,E}\right\}<\infty. \tag{5}$$

In case of number functions for $a_1=\dots=a_n=2$ and $\theta=\infty$ this theorem has been proved in [1], the necessity has been obtained also in [10], where a periodic case has been considered. In [2] it has been extended to the value $1\le\theta<\infty$ for number functions, and in case of arbitrary $a>1$ in [11]. The proof is carried out similarly and in the case of Banach- valued function. The symbol $\cong$ here (and further) means the equivalence of values (i.e.: the boundedness of value relations from above and below by positive constants not depending on f).

Remark 1. The requirement of $L_p(E)$ convergence of the series (4) in the theorem we may replace by its weak convergence. Remind that the convergence of the series $\sum_i g_i(x)\varphi(x)$ in the norm E for any function $\varphi\in S$ we call a weak convergence of the series and E- valued functions $g_i(x)$ on R^n. Prove that for $U\in UMD$, $p\in(1,\infty)^n$ as $Q^T_{[\alpha^m]}$ in [3] may stand $f_m(x,a)$, more exactly, establish the following decompositional characterization of the space $B^{\alpha}_{p,\theta}(R^n,E)$.

Theorem 2. *Let* $E\in UMD$, $\alpha=(\alpha_1,\dots,\alpha_n)>0$, $p\in(1,\infty)^n$, $1\le\theta\le\infty$. *The function* $f\in L_p(E)$ *belongs to the space* $S^{\alpha}_{p,\theta}B(R^n,E)$ *if and only if*

$$\left\{\sum_{m\ge 0}a^{[m,\alpha]\theta}\|f_m(\cdot,a)\|^{\theta}_{p,E}\right\}^{1/\theta}<\infty. \tag{6}$$

The value (6) is equivalent to the norm f *in* $S^{\alpha}_{p,\theta}B(R^n,E)$.

Necessity proof. By theorem 1 it follows, from belonging of $f \in S^{\alpha}_{p,\theta}B(R^n, E)$ that there exist a family $\left\{Q_{[a^m]}\right\}_{m\ge 0}$ of entire functions $Q_{[a^m]}(x)$ such that, the equality (4) is valid, and

$$\left\{\sum_{m\ge 0} a^{[m,\alpha]\theta}\left\|Q_{[a^m]}\right\|^{\theta}_{p,E}\right\}^{1/\theta} < \infty .$$

But, then $\hat{f} = \sum_{m\ge 0}\hat{Q}_{[a^m]}$. Therefore $\chi_{\Pi_k(a)}\hat{f} = \sum_{m\ge k}\chi_{\Pi_m(a)}\hat{Q}_{[a^m]}$ and consequently

$$f_m(x,a) = \sum_{m\ge k}\left(\hat{Q}_{[a^m]}\right)_k . \qquad (7)$$

The support $\hat{Q}_{[a^m]}$ intersects with the support $\chi_{\Pi_m(a)}$ only for $m \ge k$. From (7) for $E \in UMD$, $p \in (1,\infty)^n$ it follows that

$$\left\|f_m(\cdot,a)\right\|_{p,E} \le \sum_{m\ge k}\left\|\left(\hat{Q}_{[a^m]}\right)_k\right\|_{p,E} \le M_p\sum_{m\ge k}\left\|Q_{[a^m]}\right\|_{p,E}$$

We used the fact that the function $\chi_{\Pi_m(a)}$ is the $L_p(E)$ multiplication for $E \in UMD$, $p \in (1,\infty)^n$ (E - valued analogy of Riesz theorem [9]). Further

$$\left\{\sum_{k\ge 0} a^{[k,\alpha]\theta}\left\|f_k(\cdot,a)\right\|^{\theta}_{p,}\right\}^{1/\theta} \le M_p\left\{\sum_{k\ge 0}\left(a^{[k,\alpha]}\sum_{m\ge k}\left\|Q_{[a^m]}\right\|_{p,E}\right)^{\theta}\right\}^{1/\theta} =$$

(Minkowskii's inequality is applied)

$$= M_p\left\{\sum_{k\ge 0}\left(a^{[k,\alpha]}\sum_{l\ge 0}\left\|Q_{[a^{k+l}]}\right\|_{p,E}\right)^{\theta}\right\}^{1/\theta} \le M_p\sum_{l\ge 0}\left\{\sum_{k\ge 0}\left(a^{[k,\alpha]}\left\|Q_{[a^{k+l}]}\right\|_{p,E}\right)^{\theta}\right\}^{1/\theta} \le$$

$$\le M_p\sum_{l\ge 0}\left\{\sum_{m\ge l} a^{[m-l,\alpha]\theta}\left\|Q_{[a^m]}\right\|^{\theta}_{p,E}\right\}^{1/\theta} = M_p\sum_{l\ge 0} a^{-[l,\alpha]}\left\{\sum_{m\ge l} a^{[m,\alpha]\theta}\left\|Q_{[a^m]}\right\|^{\theta}_{p,E}\right\}^{1/\theta} \le$$

$$\le M_p C_\alpha\left\{\sum_{m\ge 0} a^{[m,\alpha]\theta}\left\|Q_{[a^m]}\right\|^{\theta}_{p,E}\right\}^{1/\theta} . \qquad (8)$$

At the last step we extended the internal summation to all $m \ge 0$ and we used $a > 1$. Since the value at the right-hand side of (8) is equivalent to the norm in $S^{\theta}_{p,\theta}B(R^n, E)$ by theorem 1, we got the estimate of the value (6) by $\|f\|_{S^{\theta}_{p,\theta}B(R^n,E)}$. Necessity is proved.

Sufficiency proof. It follows from belonging of f to $L_p(E)$, that

$$f(x) \overset{L_p(E)}{=} \sum_m f_m(x,a).$$

By virtue of theorem conditions the value (6) is finite. So, we are in application of Theorem 1. Therefore the value (6) is equivalent to the norm f in $S^{\alpha}_{p,\theta}B(R^n, E)$, and, in particular, it estimates this norm from above.

Theorem 2 is proved.

Remark 2. Note two corollaries of Theorem 2:

1. Consider that $a_1 = a_2 = \ldots = a_n = 2$. Simplify these notations, and write $f_m(x)$ instead of $f_m(x,a)$. By virtue of Theorem 2, we can take the value

$$\left\{\sum_{m\geq 0} 2^{(m,\alpha)\theta} \|f_m\|_{p,E}^{\theta}\right\}^{1/\theta} \tag{9}$$

as a norm in the space $S_{p,\theta}^{\alpha}B(R^n,E)$ for $E \in UMD$, $p \in (1,\infty)^n$.

2. Let the vector a be chosen so, that $a_1^{\alpha_1} = a_2^{\alpha_2} = \ldots = a_n^{\alpha_n} = b$. Then, the value

$$\left\{\sum_{m\geq 0} b^{|m|\theta} \left\| f_m\left(\cdot,(b)^{m:\alpha}\right)\right\|_{p,E}^{\theta}\right\}^{1/\theta}, \tag{10}$$

where $(b)^{m:\alpha} = \left(b^{m_1/\alpha_1},\ldots,b^{m_n/\alpha_n}\right)$, may serve as a norm in $S_{p,\theta}^{\alpha}B(R^n,E)$ for $E \in UMD$, $p \in (1,\infty)^n$.

Definition 1. *A totality of* E *-valued generalized functions* $f \in S'(E)$, *for which the bundles* f_m *have the properties*

$$\|f\|_{\widetilde{S}_{p,\theta}^{\alpha}B(R^n,E)} =: \left\{\sum_{m\geq 0} 2^{(m,\alpha)\theta} \|f_m\|_{p,E}^{\theta}\right\}^{1/\theta} < \infty \tag{11}$$

we call the abstract (E *-value) Besov space*

$$\widetilde{S}_{p,\theta}^{\alpha}B(R^n,E), p \in (1,\infty)^n, 1 \leq \theta \leq \infty, \alpha \in R^n.$$

It is clear that for $\alpha \in (0,\infty)^n$, $E \in UMD$ this definition coincides with before used one. In actual fact, the belonging of the function f to $S'(E)$ provides the weak convergence of the series from the bundles f_m. The convergence of these series in the norm $L_p(E)$ holds, as it was mentioned in Remark 2. Thus, $f \in L_p(E)$ and we are in conditions of Theorem 2, i.e. we return to the assumption definition (provided $E \in UMD$). It is clear that $\widetilde{S}_{p,\theta}^{\alpha}(R^n,E)$ remains as a Banach space not only for $\alpha \in (0,\infty)^n$, and for $\alpha \in R^n$ with norm given by the relation (11) (for $\alpha \in (0,\infty)^n$ this norm is equivalent to the input one).

Theorem 3. *Let* $\alpha \in (0,\infty)^n$, $E \in UMD$, $p \in (1,\infty)^n$, $1 \leq \theta \leq \infty$. *Then*

$$S_{p,\theta}^{\alpha}B(R^n,E) = \widetilde{S}_{p,\theta}^{\alpha}B(R^n,E)$$

and corresponding norms are equivalent.

References

[1]. Никольский С.М. *Функции с доминирующей смешанной производной, удовлетворяющей условию Гельдера.* Сиб.матем.ж., 1963, т.4, №6, с. 1342-1364.

[2]. Аманов Т.И. *Теоремы представления и вложения для функциональных пространств* SB. Труды МИАН СССР, 1965, т.77, с.143-167.

[3]. Джабраилов А.Д. *О некоторых функциональных пространствах.* Дан СССР, 1964, т. 159, №2.

[4]. Джабраилов М.С. *Об одном классе дифференцируемых функций нескольких переменных.* Дан Азерб. ССР, 1976, т.32, №7, с.3-7.

[5]. Шахмуров В.Б. *Теоремы вложения абстрактных функциональных пространств и их*

[3]. Джабраилов А.Д. *О некоторых функциональных пространствах.* Дан СССР, 1964, т. 159, №2.
[4]. Джабраилов М.С. *Об одном классе дифференцируемых функций нескольких переменных.* Дан Азерб. ССР, 1976, т.32, №7, с.3-7.
[5]. Шахмуров В.Б. *Теоремы вложения абстрактных функциональных пространств и их применения.* Мат.сб., 1987, т.134(176), №2, с.260-273.
[6]. Гулиев В.С. *К теории мультипликаторов интегралов Фурье для банахово- значных функций.* Труды МИРАН им. В.А. Стеклова, 1997, т. 214, с.164-181
[7]. Гулиев В.С. *Пространства Бесова банахово- значных функций.* Труды МИРАН им. В.А. Стеклова, 1997, т. 214, с. 164-181.
[8]. Джабраилов М.С. *Теоремы вложения в пространствах векторно- значных дифференцируемых функций и их приложения.* Баку, «Элм», 1996, 172 с.
[9]. Burkholder D.L. *Martingals and Fourier Analysis in Banach spaces.* Lecture Notes in Math., 1985, v. 1206, p. 61-108.
[10]. Бахвалов Н.С. *Теоремы вложения для классов функций с несколькими ограниченными производными.* Вестн. МГУ, сер. 1, математика, механика, 1963, №3, с. 7-16.
[11]. Лизоркин П.И., Никольский С.М. *Пространства функций смешанной гладкости с декомпозитной точки зрения.* Труды МИАН СССР, 1989, т. 187, с. 143-161.

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