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APPROXIMATIONAL CHARACTERISTICS OF $H^{z_0}_{\varphi,\Gamma}$ FUNCTIONS CLASSES ON QUASICONFORMAL ARCS

Abstract

The constructive characteristics of the local functions classes on quasiconformal arcs of a complex plane are obtained.

The constructive description of continuous functions classes given on quaziconformal arcs of a complex plane is obtained in the paper.

Note, that the constructive description of Holder class functions given on the piecewise-smooth boundary domain was obtained by V.K.Dzyadyk in 1959-1963 [2]-[4]. Later these results were extended to the sets of more general form in paper [9], [10], [8] and etc. We must note that the similar description is not always possible. At the same time there exist smooth arcs (see [11]) and piece-wise smooth boundary domains with zero interval and external corners, for which the direct theorems on polynomial approximation do not hold.

V.V. Andriewski has applied (see [1]) S.B.Stetchkin's approximational characteristics to obtain the constructive characteristics of H^{ω} functions classes on continuums that doesn't possess D-property (see [1]). Its essence is to attract the estimates of deviation from the functions its approximating polynomials, along with additional information on the growth of their derivatives.

The classes of $H^{z_0}_{\varphi,\Gamma}$ functions were considered in [5], [6] by using a localized continuity module.

The localized direct theorem on a polynomial approximation of functions from the class $H^{z_0}_{\varphi,\Gamma}$ on a quasiconformal arc was obtained in [7].

In this paper, the constructive characteristics of $H^{z_0}_{\varphi,\Gamma}$ class is obtained. We use a natural approximational characteristics device in this case as in V.V. Andriewskii's paper [1] to obtain the constructive characteristics of $H^{z_0}_{\varphi,\Gamma}$ functions classes by means of polynomial approximations on quasiconformal arcs.

Let Γ be a finite Jordan's arc, $\Omega = C\Gamma$ be its one-connected complement, $\infty \in \Omega$.

The function $W = \Phi(z)$ maps Ω conformally and univalently onto $\Omega' = \{W : |W| > 1$ and it is normalized by the conditions $\Phi(\infty) = \infty, \lim_{z\to\infty} \frac{1}{z}\Phi(z) > 0$.

By the same symbol Φ we denote a homeomorphism between the compactification $\widetilde{\Omega}$ of the domain Ω by prime ends by Caratheodory (see [13]), and $\overline{\Omega}'$ coinciding with $\Phi(z)$ in Ω. Each point $z \in \Gamma$ (excluding the ends of the arc Γ) in the body of prime ends $Z^{(1)}$ and $Z^{(2)} \in \widetilde{\Omega}$. Interval points $z \in \Omega$ may be also considered as the bodies of prime ends $z = Z \in \widetilde{\Omega}$. Let $\psi = \Phi^{-1}, \widetilde{\Gamma} = \widetilde{\Omega} | \Omega$ be a set of all boundary prime ends.

We shall study the quasiconformal arcs. A quasiconformal arc is the image of the segment $[-1,1]$ under some quasiconformal mapping of F plane onto itself. One of the geometrical criteria of the cuaziconformality of the arc is the following: (see [14], p.102-103) a Jordan's arc Γ is quasiconformal if and only if for three points z_1, z_2 and z_3 arranged on Γ in a growth order of indices it is valid the relation

$$|z_1 - z_2| \le c|z_1 - z_3|$$

where $c > 0$ is a constant not depending on z_1, z_2 and z_3. Denote by z_1 and z_2 the ends of the arc Γ and set for $R > 1$ and $j = 1,2$: $\Phi(z_j) = \tau_j$,

$$\Omega_1' = \{\tau : |\tau| > 1, \arg\tau_1 < \arg\tau, \arg\tau_2\},$$

$$\Omega_2' = \Omega' \overline{\Omega}_1', \Omega^j = \psi(\Omega_j'), \widetilde{\Omega}^j = \psi(\overline{\Omega}_j'),$$

$$\Gamma_R^j = \Gamma_R \cap \overline{\Omega}^j, \quad \widetilde{\Gamma}_R^j = \widetilde{\Gamma} \cap \widetilde{\Omega}^j,$$

$$\rho_R^j(z) = \inf_{\zeta \in \Gamma_R^j} |\zeta - z|, \rho_R^*(z) = \max_{j=1,2} \rho_R^j(z).$$

In papers by R.M.Rzayev [5], by J.I.Mamedkhanov and V.V.Salayev [6], the locolized continuity module was considered in the form of

$$\omega_f^{z_0}(\delta, \eta) = \sup_{\substack{|z-\tau| \le \delta \\ z, \tau \in \Gamma_\eta(z_0)}} |f(z) - f(\tau)|,$$

where $\delta, \eta > 0$ and $\Gamma_\eta(z_0) = \{z \in \Gamma : |z - z_0| \le \eta\}$.

It is obvious that if d is the diameter of Γ,

$$\omega_f^{z_0}(\delta, d) = \omega_f(\delta) = \sup_{\substack{|z-\tau| \le \delta \\ z, \tau \in \Gamma}} |f(z) - f(\tau)|$$

is the ordinary continuity module of f on Γ.

Denote by Q a class of positive functions $\varphi(\delta, \eta)$ determined for $0 < \delta, \eta < +\infty$ and such that

1. $\varphi(\delta, \eta)$ doesn't decrease in each argument;
2. $\varphi(\delta, \eta)\delta^{-1}$ doesn't increase in δ;
3. $\exists \eta \in R_+ : \lim\limits_{\delta \to 0} \varphi(\delta, \eta) = 0$;
4. $\varphi(\delta, 2\eta) \le c\varphi(\delta, \eta)$ (the constant c doesn't depend on δ and η).

Let $\varphi \in Q$ and denote

$$H_{\varphi,\Gamma}^{z_0} = \left\{ f \in C(\Gamma) : \omega_f^{z_0}(\delta, \eta) \le c\varphi(\delta, \eta), \forall \delta, \eta : 0 < \frac{\delta}{2} \le \eta \right\},$$

where $C(\Gamma)$ is a class of continuous functions on Γ.

Later we shall use the symbol $A \le B$ $(A \ge 0, B \ge 0)$, denoting that $A \le CB$, where $C > 0$ is a constant not depending on A and B.

It is valid

Theorem. *Let Γ be a quasiconformal arc. $z_0 \in \Gamma, \varphi \in Q$. For $f(z) \in H_{\varphi,\Gamma}^{z_0}$ it is necessary and sufficient the existence of succession of polynomials $\{P_n(z)\}_{n=1,2,\ldots}$, $\deg P_n \le n$, satisfying for $z \in \Gamma$ the following inequalities*

$$|f(z) - P_n(z)| \le c_1 \varphi\left(\rho_{1+\frac{1}{n}}^*(z), |z - z_0| + \rho_{1+\frac{1}{n}}^*(z) \right) \tag{1}$$

$$\left|P_n'(z)\right| \le c_2 \varphi\left(\rho^*_{1+\frac{1}{n}}(z), \left|z - z_0\right| + \rho^*_{1+\frac{1}{n}}(z)\right)\left[\rho^*_{1+\frac{1}{n}}(z)\right]^{-1}, \qquad (2)$$

where the constants $c_j, j = 1,2$ *are independent on* n, z_0, z.

Proof. Let Γ be a finite quasiconformal arc, $f(z) \in H^{z_0}_{\varphi,\Gamma}, n$ is sufficiently great. Introduce the notation $\delta(z) = \rho^*_{1+\frac{1}{n}}(z)$. In view of paper [1] the function $f(z)$ may be approximated on Γ by the trace of some function $f_n(z)$ determined and continuous in $\overline{\operatorname{int}\Gamma_3}$, continuously partially differentiable in $\Omega_3 = (\operatorname{int}\Gamma_3)/\Gamma$. By virtue of the Cauchy-Green formula

$$f_n(z) = \frac{1}{2\pi i}\int_{\Gamma_3} f_n(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\iint_{\Omega_3}\frac{\partial f_n(\zeta)}{\partial \bar{\zeta}}\frac{d\sigma_\zeta}{\varsigma - z}. \qquad (3)$$

The approximate polynomial is given by the relation

$$P_n(z) = \frac{1}{2\pi i}\int_{\Gamma_3} f_n(\zeta)\pi_n(\zeta, z)d\zeta - \frac{1}{\pi}\iint_{\Omega_3}\frac{\partial f_n(\zeta)}{\partial \bar{\zeta}}\pi_n(\zeta, z)d\sigma_\zeta, \qquad (4)$$

where $z \subset \Gamma$ and $\pi_n(\zeta, z)$ is a polynomial kernel of the form

$$\pi_n(\zeta, z) = \sum_{j=0}^{n} a_j(\zeta) z^j, n = 1,2,...$$

We have

$$\left|f(z) - P_n(z)\right| \le \left|f(z) - f_n(z)\right| + \left|f_n(z) - P_n(z)\right|. \qquad (5)$$

By using the arguments of paper [1] and the properties of $\omega^{z_0}_f(\delta, \eta)$, the estimates

$$\left|f(z) - f_n(z)\right| \le \varphi\left(\delta(z), \left|z - z_0\right| + \delta(z)\right), \qquad (6)$$

$$\left|f_n(z) - P_n(z)\right| \le \varphi\left(\delta(z), \left|z - z_0\right| + \delta(z)\right) \qquad (7)$$

are obtained.

We get from (5), (6), (7)

$$\left|f(z) - P_n(z)\right| \le \varphi\left(\delta(z), \left|z - z_0\right| + \delta(z)\right).$$

By the similar arguments of paper [1] the inequality (2) is proved.

Now, show the sufficiency of conditions (1) and (2) for $f \in H^{z_0}_{\varphi,\Gamma}$. Let δ be a sufficiently small number. We take the arbitrary points z_1 and $z_2 \in \Gamma$ with properties $\left|z_1 - z_2\right| \le \delta$. The smallest natural number for which $\rho^*_{1+\frac{1}{n}}(z_1) \le \delta$ denotes by n. Connect the points z_1 and z_2 by the arc $\gamma(z_1, z_2) \subset \Omega_1$ with the property $mes\gamma(z_1, z_2) \le \left|z_1 - z_2\right|$.

Denote

$$\eta = \max\left\{\left|z_0 - z_1\right|, \left|z_0 - z_2\right|\right\},$$

where $z_0 \in \Gamma$ is a fixed point.

The estimate

$$\left|P_n'(\zeta)\right| \le c\varphi(\delta, \eta)\delta^{-1}, \ \zeta \in \gamma(z_1, z_2)$$

holds for the polynomical satisfying the inequalities (1) and (2).

We have

$$|f(z_1)-f(z_2)|\le|f(z_1)-P_n(z_1)|+\left|\int_{\gamma(z_1,z_2)}P_n'(\xi)d\xi\right|+|P_n(z_2)-f(z_2)|\le c\varphi(\delta,\eta),$$

where c is a constant not depending on δ and η.

Hence

$$\sup_{\substack{|z_1-z_2|\le\delta \\ z_1,z_2\in O_\eta(z_0)\cap\Gamma}}|f(z_1)-f(z_2)|\le c\varphi(\delta,\eta).$$

The theorem is proved.

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