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THE INVERSE SPECTRAL PROBLEM FOR THE DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH SINGULARITY

Abstract

In this article, for given sequences $\{\lambda_n\}$ *and* $\{\mu_n\}$ *the construction of differential operator, which is a singularity, is explained. In order to accomplish this,* λ_n *and* μ_n *must have some certain asymptotics. This kind of asymptotic formulas depends on the regularity order of the function* $q(x)$. *This connection can be seen in the inverse problem which depends on the series of* $\{\lambda_n\}$ *and* $\{\mu_n\}$.

Introduction. For the regular Sturm-Liouville equations, with respect to two spectrums inverse problem completely had been solved. For those equations with the type of singularity $l(l+1)/(\pi-x)^2$, which is singular at π in the interval $[0,\pi]$, M.G.Gasimov [1], given two spectrums, has solved the inverse problem. In $[0,\pi]$, the singularity of A/x (A is real) at $x=0$ has been studied by M.G.Gasimov and R.Kh.Amirov [3] and with the help of two spectrum the solution of the inverse problem has been obtained. In $[0,\pi]$, the singularity of $\left(\frac{A}{x}+\frac{l(l+1)}{x^2}\right)$ (A is real) at $x=0$ has been studied by R.Kh.Amirov and S.Gulyaz [5] and the solution of the inverse problem with two spectrum has been given. In $[0,\pi]$, the singularity of $\frac{\delta}{x^p}$ (δ is real), $p\in(1,5/4)$ at $x=0$ has been solved. Since for $p\in(1,2)$ the eigenvalues of Sturm-Liouville operator with $\left(\frac{A}{x}+\frac{\delta}{x^p}+q(x)\right)$, $q(x)\in L_2[0,\pi]$ is potential, can not be written with single formula, for every $p\in(1,5/4)$, $p\in(5/4,3/2)$ and $p\in(3/2,2)$ cases the solution of the inverse problem is given for two spectrum by R.Kh.Amirov and Y.Chakmak [6].

In this study for Strum-Liouville operator with $\left(\frac{A}{x}+\sum_{i=1}^{3}\frac{\delta_i}{x^{p_i}}+q(x)\right)$, $q(x)\in L_2[0,\pi]$, in $[0,\pi]$ type of potential of the solution of the inverse problem is given for two spectrums

1. Expression for the normalizing constants in terms of the spectra.

We consider the differential equation

$$-y''+\left\{\frac{A}{x}+\sum_{i=1}^{3}\frac{\delta_i}{x^{p_i}}+q(x)\right\}y=\lambda y \tag{1.1}$$

and the boundary conditions

$$y(0)=0,\quad y'(\pi)-H_1y(\pi)=0, \tag{1.2}$$

$$y(0)=0,\quad y'(\pi)-H_2y(\pi)=0. \tag{1.3}$$

We assume that $q(x)\in L_2[0,\pi]$; $p_1\in(1,5/4), p_2\in(5/4,3/2), p_3\in(3/2,2), \delta_i\,(i=1.2.3)$, A, H_1 and H_2 are real numbers, with $H_1\neq H_2$. We denote the eigenvalues of the boundary value problems (1.1)-(1.2) and (1.1)-(1.3) by $\lambda_0<\lambda_1<\ldots$ and $\mu_0<\mu_1<\ldots$ respectively. Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of (1.1) satisfying the initial conditions,

$$\varphi(0,\lambda)=0,\quad \varphi'(0,\lambda)=1 \tag{1.4}$$

and

$$\psi(0,\lambda)=0,\quad \psi'(0,\lambda)=1 \tag{1.5}$$

Then it is clear that the eigenvalues $\lambda_0,\lambda_1,\ldots$ and $\mu_0,\mu_1,\ldots$ of the boundary value problems (1.1)-(1.2) and (1.1)-(1.3) coincide with the zeros of the functions
$\Phi_1(\lambda)=\varphi'(\pi,\lambda)-H_1\varphi(\pi,\lambda)$, $\Phi_2(\lambda)=\psi'(\pi,\lambda)-H_2\psi(\pi,\lambda)$.

Clearly $\varphi(x,\lambda_n)=\varphi_n(x)$ is an eigenfunction of the boundary value problem (1.1)-(1.2) and its norm is equal to

$$\alpha_n=\int_0^\pi \varphi_n^2(x)dx \tag{1.6}$$

The numbers $\alpha_0,\alpha_1,\ldots$ are called the normalizing constants of the boundary value problem (1.1)-(1.2). In this section we shall derive a formula for calculating the numbers $\alpha_0,\alpha_1,\ldots$ in terms of the two spectra $\lambda_0,\lambda_1,\ldots$ and $\mu_0,\mu_1,\ldots$.

Because $\varphi(x,\lambda)$ is the solution of the equation (1.1) that satisfying then (1.4) condition

$$\int_0^\pi \varphi^2(x,\lambda)dx=\varphi'(\pi,\lambda)\frac{\partial}{\partial\lambda}\varphi(\pi,\lambda)-\varphi(\pi,\lambda)\frac{\partial}{\partial\lambda}\varphi'(\pi,\lambda). \tag{1.7}$$

The initial conditions (1.4) and (1.5), also $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ are entire in λ for fixed x. It is clear from this that $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ are entire functions of order one half and therefore are determined by their zeros, to within a constant multiplying factor. Therefore,

$$\varphi'(\pi,\lambda)-H_1\varphi(\pi,\lambda)=C_1\prod_{k=0}^{\infty}\left(1-\frac{\lambda}{\lambda_k}\right)=\Phi_1(\lambda) \tag{1.8}$$

and

$$\psi'(\pi,\lambda)-H_2\psi(\pi,\lambda)=C_2\prod_{k=0}^{\infty}\left(1-\frac{\lambda}{\mu_k}\right)=\Phi_2(\lambda)\ , \tag{1.9}$$

where C_1 and C_2 are constants. Combining (1.10) and (1.11) we obtain

$$\int_0^\pi \varphi^2(x,\lambda_n)dx=\frac{1}{H_2-H_1}\dot{\Phi}_1(\lambda_n)\Phi_2(\lambda_n).$$

Let $\varphi_0(x,\lambda)$ be solution of equation (1.1) when $q(x)=0$ satisfying the condition (1.4). Respectively $\lambda_0^0,\lambda_1^0,\ldots$ and $\mu_0^0,\mu_1^0,\ldots$ eigenvalues problem (1.1)-(1.4) and (1.1)-(1.5) when $q(x)=0$. Then it is clear that the eigenvalues $\lambda_0^0,\lambda_1^0,\ldots$ and $\mu_0^0,\mu_1^0,\ldots$ of the boundary value problem (1.1)-(1.2) and (1.1)-(1.3) coincide with the zeros of the functions

$$\varphi_0'(\pi,\lambda)-H_1\varphi_0(\pi,\lambda)=C_1^0\prod_{k=0}^{\infty}\left(1-\frac{\lambda}{\lambda_k^0}\right)=\Phi_{01}(\lambda) \tag{1.10}$$

and

$$\psi_0'(\pi,\lambda)-H_2\psi_0(\pi,\lambda)=C_2^0\prod_{k=0}^{\infty}\left(1-\frac{\lambda}{\mu_k^0}\right)=\Phi_{02}(\lambda)\ . \tag{1.11}$$

Clearly $\varphi_0(x,\lambda_n^0)=\varphi_{0n}(x)$ is an eigenfunction of the boundary value problem (1.1)-(1.2) and its norm is equal to

$$\alpha_n^0=\int_0^{\pi}\varphi_{0n}^2(x)dx\ . \tag{1.12}$$

Because $\varphi_0(x,\lambda)$ is the solution of the equation (1.1) when $q(x)=0$ satisfying the condition (1.4)

$$\int_0^{\pi}\varphi_0^2(x,\lambda_n^0)dx=\frac{1}{H_2-H_1}\dot{\Phi}_{01}(\lambda_n^0)\Phi_{02}(\lambda_n^0).$$

Therefore

$$\alpha_n=\frac{1}{H_2-H_1}\dot{\Phi}_1(\lambda_n)\Phi_2(\lambda_n), \tag{1.13}$$

and

$$\alpha_n^0=\frac{1}{H_2-H_1}\dot{\Phi}_{01}(\lambda_n^0)\Phi_{02}(\lambda_n^0), \tag{1.14}$$

Then we obtain

$$\dot{\Phi}_1(\lambda_n)=-\frac{C_1}{\lambda_n}\prod_{k=0}^{\infty}{}'\left(1-\frac{\lambda_n}{\lambda_k}\right),\quad \dot{\Phi}_{01}(\lambda_n^0)=-\frac{C_1^0}{\lambda_n^0}\prod_{k=0}^{\infty}{}'\left(1-\frac{\lambda_n^0}{\lambda_k^0}\right)$$

where (here and henceforth) the symbol Π' means that the factor $k=n$ is excluded from the infinite product. These formula and (1.13)-(1.14) imply that

$$\alpha_n=-\frac{C_1C_2}{H_2-H_1}\frac{1}{\lambda_n}\prod_{k=0}^{\infty}{}'\left(1-\frac{\lambda_n}{\lambda_k}\right)\prod_{k=0}^{\infty}\left(1-\frac{\lambda_n}{\mu_k}\right), \tag{1.15}$$

and

$$\alpha_n^0=-\frac{C_1^0C_2^0}{H_2-H_1}\frac{1}{\lambda_n^0}\prod_{k=0}^{\infty}{}'\left(1-\frac{\lambda_n^0}{\lambda_k^0}\right)\prod_{k=0}^{\infty}\left(1-\frac{\lambda_n^0}{\mu_k^0}\right), \tag{1.16}$$

Since

$$\Phi_2(\lambda_n)=C_2\prod_{k=0}^{\infty}\left(1-\frac{\lambda_n}{\mu_k}\right)=B_2C_2\prod_{k=0}^{\infty}\left(1-\frac{\lambda_n^0}{\mu_k^0}\right)\prod_{k=0}^{\infty}\frac{\lambda_n-\mu_k^0}{\lambda_n^0-\mu_k^0}\prod_{k=0}^{\infty}\frac{\lambda_n-\mu_k}{\lambda_n-\mu_k^0}.$$

In the same way we can prove that

$$\dot{\Phi}_1(\lambda_n)=-\frac{B_1C_1}{\lambda_n^0}\prod_{k=0}^{\infty}{}'\left(\frac{\lambda_n^0}{\lambda_k^0}\right)\prod_{k=0}^{\infty}{}'\left(\frac{\lambda_n-\lambda_k}{\lambda_n^0-\lambda_k^0}\right)\prod_{k=0}^{\infty}{}'\left(1-\frac{\lambda_n-\lambda_k}{\lambda_n-\lambda_k^0}\right).$$

These formula and (1.13) imply that

$$\alpha_n=\frac{C_1C_2}{C_1^0C_2^0}B_1B_2\alpha_n^0\frac{\lambda_n-\mu_n}{\lambda_n^0-\mu_n^0}\prod_{k=0}^{\infty}{}'\left(\frac{\lambda_n-\lambda_k^0}{\lambda_n^0-\lambda_k^0}\right)\prod_{k=0}^{\infty}{}'\left(\frac{\lambda_n-\lambda_k}{\lambda_n-\lambda_k^0}\right)\prod_{k=0}^{\infty}{}'\left(\frac{\lambda_n-\mu_k^0}{\lambda_n^0-\mu_k^0}\right)\prod_{k=0}^{\infty}{}'\left(\frac{\lambda_n-\mu_k}{\lambda_n-\mu_k^0}\right)$$

It can be shown easily that

$$\frac{C_1C_2}{C_1^0C_2^0}B_1B_2=1. \tag{1.17}$$

Therefore

$$\alpha_n = \alpha_n^0 \frac{\lambda_n - \mu_n}{\lambda_n^0 - \mu_n^0} \frac{\lambda_n - \lambda_0}{\lambda_n^0 - \lambda_0^0} \frac{\lambda_n - \mu_0}{\lambda_0^0 - \mu_0^0} \times$$

$$\times \prod_{k=1}^{\infty}{}' \left(\frac{\lambda_n - \lambda_k^0}{\lambda_n^0 - \lambda_k^0} \right) \prod_{k=1}^{\infty}{}' \left(\frac{\lambda_n - \lambda_k}{\lambda_n - \lambda_k^0} \right) \prod_{k=1}^{\infty}{}' \left(\frac{\lambda_n - \mu_k^0}{\lambda_n^0 - \mu_k^0} \right) \prod_{k=1}^{\infty}{}' \left(\frac{\lambda_n - \mu_k}{\lambda_n - \mu_k^0} \right). \quad (1.18)$$

2. Asymptotic formulas for the numbers α_n.

Suppose that we are given two sequences of numbers $\lambda_0, \lambda_1, \dots, \mu_0, \mu_1, \dots,$ and that we known them to be the eigenvalues of the boundary value problems (1.1)-(1.2) and (1.1)-(1.3) with an unknown function $q(x)$ and with the real numbers H_1, H_2. Suppose that the following asymptotic formulas hold:

$$\lambda_n = (n+1/2)^2 + \sum_{i=0}^{2} a_{1i} (n+1/2)^{p_{3-i}-1} + a_3 (n+1/2)^{2p_3-3} + \frac{A}{\pi} \ln(n+1/2) + a_8 (n+1/2)^{3p_3-5} +$$
$$+ a_{11} (n+1/2)^{4p_3-7} + a_{14} (n+1/2)^{5p_3-9} + \sum_{i=0}^{2} a_{2i} + a_{43} \frac{\ln(n+1/2)}{(n+1/2)^{2-p_3}} + a_{53} \frac{1}{(n+1/2)^{2-p_3}} +$$
$$+ a_{32} \frac{1}{(n+1/2)^{3-2p_2}} + a_{93} \frac{\ln(n+1/2)}{(n+1/2)^{4-2p_3}} + \gamma_{13} \frac{1}{(n+1/2)^{4-2p_3}} + a_{123} \frac{\ln(n+1/2)}{(n+1/2)^{6-3p_3}} +$$
$$+ \gamma_{73} \frac{1}{(n+1/2)^{6-3p_3}} + a_{31} \frac{1}{(n+1/2)^{3-2p_1}} + a_{41} \frac{\ln(n+1/2)}{(n+1/2)^{2-p_1}} + a_{51} \frac{1}{(n+1/2)^{2-p_1}} +$$
$$+ a_{82} \frac{1}{(n+1/2)^{5-3p_2}} + a_{42} \frac{\ln(n+1/2)}{(n+1/2)^{2-p_2}} + a_{52} \frac{1}{(n+1/2)^{2-p_2}} + a_6 \frac{\ln(n+1/2)}{(n+1/2)} + a_7 \frac{1}{(n+1/2)} +$$
$$+ a_{92} \frac{\ln(n+1/2)}{(n+1/2)^{4-2p_3}} + \gamma_{12} \frac{1}{(n+1/2)^{4-2p_2}} + a_{103} \frac{\ln^2(n+1/2)}{(n+1/2)^{3-p_3}} + \gamma_{23} \frac{\ln(n+1/2)}{(n+1/2)^{3-p_3}} +$$
$$+ \gamma_{33} \frac{1}{(n+1/2)^{3-p_3}} + a_{133} \frac{\ln^2(n+1/2)}{(n+1/2)^{5-2p_3}} + \gamma_{83} \frac{\ln(n+1/2)}{(n+1/2)^{5-2p_3}} + \gamma_{93} \frac{1}{(n+1/2)^{5-2p_3}} +$$
$$+ a_{83} \frac{1}{(n+1/2)^{5-3p_1}} + a_{102} \frac{\ln^2(n+1/2)}{(n+1/2)^{3-p_2}} + \gamma_{22} \frac{\ln(n+1/2)}{(n+1/2)^{3-p_2}} + \gamma_{32} \frac{1}{(n+1/2)^{3-p_2}} +$$
$$+ a_{91} \frac{\ln(n+1/2)}{(n+1/2)^{4-2p_1}} + \gamma_{11} \frac{1}{(n+1/2)^{4-2p_1}} + a_{112} \frac{1}{(n+1/2)^{7-4p_2}} + a_{101} \frac{\ln^2(n+1/2)}{(n+1/2)^{3-p_1}} +$$
$$+ \gamma_{21} \frac{\ln(n+1/2)}{(n+1/2)^{3-p_1}} + \gamma_{31} \frac{1}{(n+1/2)^{3-p_1}} + a_{22} \frac{\ln(n+1/2)}{(n+1/2)^{6-3p_2}} + \gamma_{72} \frac{1}{(n+1/2)^{6-3p_2}} -$$
$$- \frac{A^3}{12\pi} \frac{\ln^3(n+1/2)}{(n+1/2)^2} + \gamma_4 \frac{\ln^2(n+1/2)}{(n+1/2)^2} + \gamma_5 \frac{\ln(n+1/2)}{(n+1/2)^2} + \gamma_6 \frac{1}{(n+1/2)^2} + O\left(\frac{\ln^3 n}{n^{4-p_3}} \right) \quad (2.1)$$

and

$$\mu_n = (n+1/2)^2 + \sum_{i=0}^{2} a'_{1i} (n+1/2)^{p_{3-i}-1} + a'_3 (n+1/2)^{2p_3-3} + \frac{A}{\pi} \ln(n+1/2) + a'_8 (n+1/2)^{3p_3-5} +$$
$$+ a'_{11} (n+1/2)^{4p_3-7} + a'_{14} (n+1/2)^{5p_3-9} + \sum_{i=0}^{2} a'_{2i} + a'_{43} \frac{\ln(n+1/2)}{(n+1/2)^{2-p_3}} + a'_{53} \frac{1}{(n+1/2)^{2-p_3}} +$$

$$+a'_{32}\frac{1}{(n+1/2)^{3-2p_2}}+a'_{93}\frac{\ln(n+1/2)}{(n+1/2)^{4-2p_3}}+\gamma'_{13}\frac{1}{(n+1/2)^{4-2p_3}}+a'_{123}\frac{\ln(n+1/2)}{(n+1/2)^{6-3p_3}}+$$

$$+\gamma'_{73}\frac{1}{(n+1/2)^{6-3p_3}}+a'_{31}\frac{1}{(n+1/2)^{3-2p_1}}+a'_{41}\frac{\ln(n+1/2)}{(n+1/2)^{2-p_1}}+a'_{51}\frac{1}{(n+1/2)^{2-p_1}}+$$

$$+a'_{82}\frac{1}{(n+1/2)^{5-3p_2}}+a'_{42}\frac{\ln(n+1/2)}{(n+1/2)^{2-p_2}}+a'_{52}\frac{1}{(n+1/2)^{2-p_2}}+a'_{6}\frac{\ln(n+1/2)}{(n+1/2)}+a'_{7}\frac{1}{(n+1/}$$

$$+a'_{92}\frac{\ln(n+1/2)}{(n+1/2)^{4-2p_2}}+\gamma'_{12}\frac{1}{(n+1/2)^{4-2p_2}}+a'_{103}\frac{\ln^2(n+1/2)}{(n+1/2)^{3-p_3}}+\gamma'_{23}\frac{\ln(n+1/2)}{(n+1/2)^{3-p_3}}+$$

$$+\gamma'_{33}\frac{1}{(n+1/2)^{3-p_3}}+a'_{133}\frac{\ln^2(n+1/2)}{(n+1/2)^{5-2p_3}}+\gamma'_{83}\frac{\ln(n+1/2)}{(n+1/2)^{5-2p_3}}+\gamma'_{93}\frac{1}{(n+1/2)^{5-2p_3}}+$$

$$+a'_{83}\frac{1}{(n+1/2)^{5-3p_3}}+a'_{102}\frac{\ln^2(n+1/2)}{(n+1/2)^{3-p_2}}+\gamma'_{22}\frac{\ln(n+1/2)}{(n+1/2)^{3-p_2}}+\gamma'_{32}\frac{1}{(n+1/2)^{3-p_2}}+$$

$$+a'_{91}\frac{\ln(n+1/2)}{(n+1/2)^{4-2p_1}}+\gamma'_{11}\frac{1}{(n+1/2)^{4-2p_1}}+a'_{112}\frac{1}{(n+1/2)^{7-4p_2}}+a'_{101}\frac{\ln^2(n+1/2)}{(n+1/2)^{3-p_1}}+$$

$$+\gamma'_{21}\frac{\ln(n+1/2)}{(n+1/2)^{3-p_1}}+\gamma'_{31}\frac{1}{(n+1/2)^{3-p_1}}+a'_{22}\frac{\ln(n+1/2)}{(n+1/2)^{6-3p_2}}+\gamma'_{72}\frac{1}{(n+1/2)^{6-3p_2}}-$$

$$-\frac{A^3}{12\pi}\frac{\ln^3(n+1/2)}{(n+1/2)^2}+\gamma'_4\frac{\ln^2(n+1/2)}{(n+1/2)^2}+\gamma'_5\frac{\ln(n+1/2)}{(n+1/2)^2}+\gamma'_6\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^3 n}{n^{4-p_3}}\right)\quad (2$$

Then, by (2.1) and (2.2)

$$a_{2i}=\frac{1}{\pi}\left[-H_1+\frac{A\ln\pi}{2}+AM_5+\frac{\delta_i}{2(1-p_i)\pi^{p_i-1}}+\frac{1}{2}\int_0^\pi q(t)dt\right],\qquad (i=1,2,3)$$

$$a'_{2i}=\frac{1}{\pi}\left[-H_2+\frac{A\ln\pi}{2}+AM_5+\frac{\delta_i}{2(1-p_i)\pi^{p_i-1}}+\frac{1}{2}\int_0^\pi q(t)dt\right],\qquad (i=1,2,3).$$

Therefore

$$H_1-H_2=\pi(a'_{2i}-a_{2i})\,.\qquad (2$$

It follows from (2.1) and (2.2) that by hypothesis, the numbers $\{\lambda_n\}$ and $\{\mu_n\}$ are dist spectra of one and the same equation. Hence it follows from the result of the prev section that the normalizing constants of the boundary value problem (2.1)-(2.2) are g by the formula (1.18).

In the rest of this section we shall find the asymptotic formula for the numbers in terms of the known asymptotic formulae for the numbers λ_n and μ_n. We shall c out this derivation in several stages. We first consider the infinite product $\psi(\lambda$ $=\prod_{k=1}^{\infty}\left(1+\frac{\lambda_k-\lambda_k^0}{\lambda_k^0-\lambda_n}\right)$. Clearly $\ln\psi(\lambda_n)=\sum_{k=1}^{\infty}{}' \ln\left(1+\frac{\lambda_k-\lambda_k^0}{\lambda_k^0-\lambda_n}\right)$. For sufficiently large n

$k\neq n\left|\frac{\lambda_k-\lambda_k^0}{\lambda_k^0-\lambda_n}\right|<\frac{C}{n+1/2}$, (in what follows C will denote a constant, not necessarily same one). Therefore

$$\ln\psi(\lambda_n) = -\sum_{k=1}^{\infty}{}' \left[\sum_{r=1}^{\infty} \frac{(-1)^r}{r} \left(\frac{\lambda_k - \lambda_k^0}{\lambda_k^0 - \lambda_n} \right)^r \right] . \tag{2.4}$$

We have

Lemma. *Let* $\left|\lambda_k - \lambda_k^0\right| \le a \quad (k = 1,2,\ldots)$. *Then*

$$\sum_{k=1}^{\infty}{}' \left| \frac{\lambda_k - \lambda_k^0}{\lambda_k^0 - \lambda_n} \right|^r \le \begin{cases} C\dfrac{\ln(n+1/2)}{(n+1/2)}, & r = 1 \\ C\dfrac{a^r}{(n+1/2)^r}, & r \ge 2 \end{cases} \tag{2.5}$$

The proof of Lemma is same as the proof of [3].

Let us study $\ln\psi(\lambda_n)$ a little further. It follows from the estimate (2.7) that

$$\left| \sum_{k=1}^{\infty}{}' \sum_{r=3}^{\infty} \frac{(-1)^r}{r} \left(\frac{\lambda_k - \lambda_k^0}{\lambda_k^0 - \lambda_n} \right)^r \right| \le \sum_{r=3}^{\infty} \frac{1}{r} \sum_{k=1}^{\infty}{}' \left| \frac{\lambda_k - \lambda_k^0}{\lambda_k^0 - \lambda_n} \right|^r$$
$$\le C \sum_{r=3}^{\infty} \frac{a^r}{(n+1/2)^r} = \frac{Ca^3}{(n+1/2)^3} \sum_{r=0}^{\infty} \frac{a^r}{(n+1/2)^r} = O\left(\frac{1}{n^3}\right) \tag{2.6}$$

This estimate and the formula (2.6) show that

$$\ln\psi(\lambda_n) = \sum_{k=1}^{\infty}{}' \frac{\lambda_k - \lambda_k^0}{\lambda_k^0 - \lambda_n} - \frac{1}{2} \sum_{k=1}^{\infty}{}' \left(\frac{\lambda_k - \lambda_k^0}{\lambda_k^0 - \lambda_n} \right)^2 + O\left(\frac{1}{n^3}\right) \tag{2.7}$$

Now we consider the behavior of the sums appearing in this formula.

Using the asymptotic formula (2.1) and (2.2) we see that the first sum is satisfied. If we use expression λ_k and λ_k^0 asymptotic, we get the expression

$$\sum_{k=1}^{\infty}{}' \frac{\lambda_k - \lambda_k^0}{\lambda_k^0 - \lambda_n} = \frac{1}{3} \sum_{i=0}^{2} \left(\pi^2 a_{1i} - 6a_{1i}^0\right)\left(a_{2i} - a_{2i}^0\right) \frac{1}{(n+1/2)^{3-p_{3-i}}} + \left(\gamma_{93} - \gamma_{93}^0\right) \frac{1}{(n+1/2)^{5-2p_3}} +$$
$$+ \frac{A\left(3\pi^2 - 4\right)}{12\pi} \sum_{i=1}^{3} \left(a_{2i} - a_{2i}^0\right) \frac{\ln(n+1/2)}{(n+1/2)^2} + \left[M_\lambda + \sum_{i=1}^{3} \frac{\left(a_{2i} - a_{2i}^0\right)}{2} \left(3 + \frac{\pi^2\left(a_{2i} - a_{2i}^0\right)}{6} + \frac{2A\ln(3/2)}{3\pi} \right) + \right.$$
$$\left. + \sum_{i=1}^{3} \left(\left(\frac{3}{2}\right)^{p_i - 1} \left(a_{5i} - a_{5i}^0\right) + \frac{\gamma_{1i} - \gamma_{1i}^0}{2^{2p_i - 2} 3^{3 - 2p_i}} + \ln(3/2) \frac{\gamma_{2i} - \gamma_{2i}^0}{2^{p_i - 1} 3^{2 - p_i}} + \frac{\gamma_{3i} - \gamma_{3i}^0}{2^{p_i - 1} 3^{2 - p_i}} \right) \right] \frac{1}{(n+1/2)^2} +$$
$$+ O\left(\frac{\ln^3 n}{n^{4 - p_3}} \right) \tag{2.8}$$

$$-\frac{1}{2} \sum_{k=1}^{\infty}{}' \left(\frac{\lambda_k - \lambda_k^0}{\lambda_k^0 - \lambda_n} \right) = \frac{\pi^2}{4} \sum_{i=1}^{3} \left(a_{2i} - a_{2i}^0\right)^2 \frac{1}{(n+1/2)^2} + O\left(\frac{1}{n^4}\right) . \tag{2.9}$$

Therefore, the formulae (2.7), (2.8) and (2.9) imply that

$$\ln\psi(\lambda_n)=\sum_{k=1}^{\infty}{}' \frac{\lambda_k-\lambda_k^0}{\lambda_k^0-\lambda_n}-\frac{1}{2}\sum_{k=1}^{\infty}{}'\left(\frac{\lambda_k-\lambda_k^0}{\lambda_k^0-\lambda_n}\right)^2+O\left(\frac{1}{n^3}\right)=$$
$$=\frac{1}{3}\sum_{i=0}^{2}\left(\pi^2 a_{1i}-6a_{1i}^0\right)\left(a_{2i}-a_{2i}^0\right)\frac{1}{(n+1/2)^{3-p_{3-i}}}+\left(\gamma_{93}-\gamma_{93}^0\right)\frac{1}{(n+1/2)^{5-2p_3}}+$$
$$+\frac{A(3\pi^2-4)}{12\pi}\sum_{i=1}^{3}\left(a_{2i}-a_{2i}^0\right)\frac{\ln(n+1/2)}{(n+1/2)^2}+\left[M_\lambda+\sum_{i=1}^{3}\frac{\left(a_{2i}-a_{2i}^0\right)}{2}\left(3+\frac{\pi^2\left(a_{2i}-a_{2i}^0\right)}{6}+\right.\right.$$
$$\left.+\frac{2A\ln(3/2)}{3\pi}\right)+\sum_{i=1}^{3}\left(\left(\frac{3}{2}\right)^{p_i-1}\left(a_{5i}-a_{5i}^0\right)+\frac{\gamma_{1i}-\gamma_{1i}^0}{2^{2p_i-2}3^{3-2p_i}}+\ln(3/2)\frac{\gamma_{2i}-\gamma_{2i}^0}{2^{p_i-1}3^{2-p_i}}+\right.$$
$$\left.\left.+\frac{\gamma_{3i}-\gamma_{3i}^0}{2^{p_i-1}3^{2-p_i}}\right)+\frac{\pi^2}{4}\sum_{i=1}^{3}\left(a_{2i}-a_{2i}^0\right)^2\right]\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^3 n}{n^{4-p_3}}\right)$$

Hence

$$\psi(\lambda_n)=1+\frac{1}{3}\sum_{i=0}^{2}\left(\pi^2 a_{1i}-6a_{1i}^0\right)\left(a_{2i}-a_{2i}^0\right)\frac{1}{(n+1/2)^{3-p_{3-i}}}+\left(\gamma_{93}-\gamma_{93}^0\right)\frac{1}{(n+1/2)^{5-2p_3}}+$$
$$+\frac{A(3\pi^2-4)}{12\pi}\sum_{i=1}^{3}\left(a_{2i}-a_{2i}^0\right)\frac{\ln(n+1/2)}{(n+1/2)^2}+\left[M_\lambda+\sum_{i=1}^{3}\frac{\left(a_{2i}-a_{2i}^0\right)}{2}\left(3+\frac{\pi^2\left(a_{2i}-a_{2i}^0\right)}{6}+\right.\right.$$
$$\left.+\frac{2A\ln(3/2)}{3\pi}\right)+\sum_{i=1}^{3}\left(\left(\frac{3}{2}\right)^{p_i-1}\left(a_{5i}-a_{5i}^0\right)+\frac{\gamma_{1i}-\gamma_{1i}^0}{2^{2p_i-2}3^{3-2p_i}}+\ln(3/2)\frac{\gamma_{2i}-\gamma_{2i}^0}{2^{p_i-1}3^{2-p_i}}+\right. \tag{2.10}$$
$$\left.\left.+\frac{\gamma_{3i}-\gamma_{3i}^0}{2^{p_i-1}3^{2-p_i}}\right)+\frac{\pi^2}{4}\sum_{i=1}^{3}\left(a_{2i}-a_{2i}^0\right)^2\right]\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^3 n}{n^{4-p_3}}\right)$$

Now we consider the behavior of the infinity products

$$\prod_{k=1}^{\infty}{}'\left(\frac{\lambda_n-\mu_k}{\lambda_n-\mu_k^0}\right)=1+\frac{1}{3}\sum_{i=0}^{2}\left(\pi^2 a'_{1i}-6a'^0_{1i}\right)\left(a'_{2i}-a'^0_{2i}\right)\frac{1}{(n+1/2)^{3-p_{3-i}}}+\left(\gamma'_{93}-\gamma'^0_{93}\right)\frac{1}{(n+1/2)^{5-2p_3}}+$$
$$+\frac{A(3\pi^2-4)}{12\pi}\sum_{i=1}^{3}\left(a'_{2i}-a'^0_{2i}\right)\frac{\ln(n+1/2)}{(n+1/2)^2}+\left[M_\lambda+\sum_{i=1}^{3}\frac{\left(a'_{2i}-a'^0_{2i}\right)}{2}\left(3+\frac{\pi^2\left(a'_{2i}-a'^0_{2i}\right)}{6}+\right.\right.$$
$$\left.+\frac{2A\ln(3/2)}{3\pi}\right)+\sum_{i=1}^{3}\left(\left(\frac{3}{2}\right)^{p_i-1}\left(a'_{5i}-a'^0_{5i}\right)+\frac{\gamma'_{1i}-\gamma'^0_{1i}}{2^{2p_i-2}3^{3-2p_i}}+\ln(3/2)\frac{\gamma'_{2i}-\gamma'^0_{2i}}{2^{p_i-1}3^{2-p_i}}+\right.$$
$$\left.\left.+\frac{\gamma'_{3i}-\gamma'^0_{3i}}{2^{p_i-1}3^{2-p_i}}\right)+\frac{\pi^2}{4}\sum_{i=1}^{3}\left(a'_{2i}-a'^0_{2i}\right)^2\right]\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^3 n}{n^{4-p_3}}\right) \tag{2.11}$$

here

$$M_\mu=\sum_{k=1}^{\infty}\left\{\mu_k-\mu_k^0-\sum_{i=1}^{3}\left[2\left(a'_{2i}-a'^0_{2i}\right)-\frac{2\left(a'_{5i}-a'^0_{5i}\right)}{(k+1/2)^{2-p_i}}-\frac{\left(\gamma'_{1i}-\gamma'^0_{1i}\right)}{(k+1/2)^{4-2p_i}}-\right.\right.$$
$$\left.\left.-\left(\gamma'_{2i}-\gamma'^0_{2i}\right)\frac{\ln(k+1/2)}{(k+1/2)^{3-p_i}}-\frac{\left(\gamma'_{3i}-\gamma'^0_{3i}\right)}{(k+1/2)^{3-p_i}}\right]\right\}.$$

In the same way we can prove that

$$\psi\left(\lambda_n^0\right)=\prod_{k=1}^{\infty}{}'\left(\frac{\lambda_n-\lambda_k^0}{\lambda_n^0-\lambda_k^0}\right)=1+\frac{1}{3}\sum_{i=1}^{3}\left(\pi^2 a_{1i}-6a_{1i}^0\right)\left(a_{2i}-a_{2i}^0\right)\frac{1}{(n+1/2)^{3-p_1}}+$$
$$+\left(\gamma_{93}-\gamma_{93}^0\right)\frac{1}{(n+1/2)^{5-2p_3}}+\frac{A\left(3\pi^2-4\right)}{12}\sum_{i=1}^{3}\left(a_{2i}-a_{2i}^0\right)\frac{\ln(n+1/2)}{(n+1/2)^2}+$$
$$+\frac{1}{2}\sum_{i=1}^{3}\left[\left(a_{2i}-a_{2i}^0\right)\left(3+\frac{\pi^2\left(a_{2i}-a_{2i}^0\right)}{6}+\frac{2A\ln(3/2)}{3\pi}\right)-\right.$$
$$\left.-\frac{\pi^2\left(a_{2i}-a_{2i}^0\right)^2}{4}\right]\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^3}{n^{4-p_3}}\right) \tag{2.12}$$

$$\prod_{k=1}^{\infty}{}'\left(\frac{\lambda_n-\mu_k^0}{\lambda_n^0-\mu_k^0}\right)=1+\frac{1}{3}\sum_{i=1}^{3}\left(\pi^2 a'_{1i}-6a'^0_{1i}\right)\left(a'_{2i}-a'^0_{2i}\right)\frac{1}{(n+1/2)^{3-p_1}}+\left(\gamma'_{93}-\gamma'^0_{93}\right)\frac{1}{(n+1/2)^{5-2p_3}}+$$
$$+\frac{A\left(3\pi^2-4\right)}{12}\sum_{i=1}^{3}\left(a'_{2i}-a'^0_{2i}\right)\frac{\ln(n+1/2)}{(n+1/2)^2}+\frac{1}{2}\sum_{i=1}^{3}\left[\left(a'_{2i}-a'^0_{2i}\right)\left(3+\frac{\pi^2\left(a'_{2i}-a'^0_{2i}\right)}{6}+\frac{2A\ln(3/2)}{3\pi}\right)-\right.$$
$$\left.-\frac{\pi^2\left(a'_{2i}-a'^0_{2i}\right)^2}{4}\right]\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^3}{n^{4-p_3}}\right). \tag{2.13}$$

It is clear from the asymptotic formulas (2.1) and (2.2) that

$$\frac{\lambda_n-\mu_n}{\lambda_n^0-\mu_n^0}=1+\sum_{i=1}^{3}\left[\frac{a_{5i}-a'_{5i}}{a_{2i}-a'_{2i}}\frac{a_{5i}^0-a'^0_{5i}}{a_{2i}^0-a'^0_{2i}}\right]\frac{1}{(n+1/2)^{4-2p_1}}+$$
$$+\sum_{i=1}^{3}\left[\frac{a_{5i}-a'_{5i}}{a_{2i}-a'_{2i}}\frac{a_{10i}^0-a'^0_{10i}}{a_{2i}^0-a'^0_{2i}}+\frac{a_{10i}-a'_{10i}}{a_{2i}-a'_{2i}}\frac{a_{5i}^0-a'^0_{5i}}{a_{2i}^0-a'^0_{2i}}\right]\frac{1}{(n+1/2)^{3-p_1}}+$$
$$+\sum_{i=1}^{3}\left\{\frac{1}{2}\left[\left(a_{2i}+a'_{2i}\right)+\left(a_{2i}^0+a'^0_{2i}\right)\right]+\frac{a_{10i}-a'_{10i}}{a_{2i}-a'_{2i}}\frac{a_{10i}^0-a'^0_{10i}}{a_{2i}^0-a'^0_{2i}}\right\}\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^3}{n^{4-p_3}}\right) \tag{2.14}$$

$$\frac{\lambda_n-\lambda_0}{\lambda_n^0-\lambda_0^0}=1+\sum_{i=1}^{3}\frac{2\left(a_{1i}-a_{1i}^0\right)}{(n+1/2)^{3-p_1}}+\left[\sum_{i=1}^{3}2\left(a_{2i}-a_{2i}^0\right)+\left(\lambda_0^0-\lambda_0\right)\right]\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^3}{n^{4-p_3}}\right) \tag{2.15}$$

$$\frac{\lambda_n-\mu_0}{\lambda_n^0-\mu_0^0}=1+\sum_{i=1}^{3}\frac{2\left(a_{1i}-a_{1i}^0\right)}{(n+1/2)^{3-p_1}}+\left[\sum_{i=1}^{3}2\left(a_{2i}-a_{2i}^0\right)+\left(\mu_0^0-\mu_0\right)\right]\frac{1}{(n+1/2)^2}+O\left(\frac{\ln^3}{n^{4-p_3}}\right) \tag{2.16}$$

It is well known that

$$a_n^0=\frac{\pi}{2}+\frac{M_4\pi}{2}\sum_{i=1}^{3}\frac{\delta_i}{(n+1/2)^{2-p_1}}+\frac{\nu_{73}^0}{(n+1/2)^{6-3p_3}}+\frac{\nu_{13}^0}{(n+1/2)^{4-2p_3}}+\frac{AN\pi}{2}\frac{1}{(n+1/2)}+$$
$$+\nu_{23}^0\frac{\ln(n+1/2)}{(n+1/2)^{3-p_3}}+\frac{\nu_{12}^0}{(n+1/2)^{4-2p_2}}+\nu_{22}^0\frac{\ln(n+1/2)}{(n+1/2)^{3-p_2}}+\frac{\nu_{32}^0}{(n+1/2)^{3-p_2}}+$$
$$+\nu_{83}^0\frac{\ln(n+1/2)}{(n+1/2)^{5-2p_3}}+\frac{\nu_{11}^0}{(n+1/2)^{4-2p_1}}+\frac{\nu_{72}^0}{(n+1/2)^{6-3p_2}}+\nu_{21}^0\frac{\ln(n+1/2)}{(n+1/2)^{3-p_1}}+$$
$$+\nu_4^0\frac{\ln^2(n+1/2)}{(n+1/2)^2}+\nu_5^0\frac{\ln(n+1/2)}{(n+1/2)^2}+\frac{\nu_6^0}{(n+1/2)^2}+O\left(\frac{\ln^2 n}{n^{4-p_3}}\right). \tag{2.17}$$

Therefore we obtain the formulas (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), (2.16) and (2.17) the following formula for the numbers α_n:

$$
\begin{aligned}
\alpha_n = {} & \frac{\pi}{2} + \frac{M_4\pi}{2}\sum_{i=1}^{3}\frac{\delta_i}{(n+1/2)^{2-p_i}} + \left[v_{13}^0 + \frac{\pi}{2}\frac{a_{53}-a'_{53}}{a_{23}-a'_{23}}\frac{a_{53}^0-a'^0_{53}}{a_{23}^0-a'^0_{23}}\right]\frac{1}{(n+1/2)^{4-2p_3}} + \\
& + \frac{AN\pi}{2}\frac{1}{(n+1/2)} + \left[v_{12}^0 + \frac{\pi}{2}\frac{a_{52}-a'_{52}}{a_{22}-a'_{22}}\frac{a^0_{52}-a'^0_{52}}{a^0_{22}-a'^0_{22}}\right]\frac{1}{(n+1/2)^{4-2p_2}} + v_{23}^0\frac{\ln(n+1/2)}{(n+1/2)^{3-p_3}} + \\
& + \Big\{2\pi(a_{13}-a_{13}^0) + \pi(a_{23}-a_{23}^0) + \left(\frac{\pi^2 a_{13}-6a_{13}^0}{3}\right) + \pi(a'_{23}-a'^0_{23})\left(\frac{\pi^2 a'_{13}-6a'^0_{13}}{3}\right) \times \\
& \times \frac{\pi}{2}\frac{a_{53}-a'_{53}}{a_{23}-a'_{23}}\frac{a^0_{103}-a'^0_{103}}{a^0_{23}-a'^0_{23}} + \frac{\pi}{2}\frac{a_{103}-a'_{103}}{a_{23}-a'_{23}}\frac{a^0_{53}-a'^0_{53}}{a^0_{23}-a'^0_{23}} + v^0_{33}\Big\}\frac{1}{(n+1/2)^{3-p_3}} \times \\
& \times \left[v^0_{11} + \frac{\pi}{2}\frac{a_{51}-a'_{51}}{a_{21}-a'_{21}}\frac{a^0_{51}-a'^0_{51}}{a^0_{21}-a'^0_{21}}\right]\frac{1}{(n+1/2)^{4-2p_1}} + \sum_{i=1}^{3}\Big[v_{2i}\frac{\ln(n+1/2)}{(n+1/2)^{3-p_i}} + \\
& + \Big\{2\pi(a_{1i}-a^0_{1i}) + \pi(a_{2i}-a^0_{2i})\left(\frac{\pi^2 a_{1i}-6a^0_{1i}}{3}\right) + \pi(a'_{2i}-a'^0_{2i})\left(\frac{\pi^2 a'_{1i}-6a'^0_{1i}}{3}\right) + \\
& + \frac{\pi}{2}\frac{a_{5i}-a'_{5i}}{a_{2i}-a'_{2i}}\frac{a^0_{10i}-a'^0_{10i}}{a^0_{2i}-a'^0_{2i}} + \frac{\pi}{2}\frac{a_{10i}-a'_{10i}}{a_{2i}-a'_{2i}}\frac{a^0_{5i}-a'^0_{5i}}{a^0_{2i}-a'^0_{2i}} + v^0_{3i}\Big\}\frac{1}{(n+1/2)^{3-p_i}}\Big] + v^0_4\frac{\ln^2(n+1/2)}{(n+1/2)^2} + \\
& + \sum_{i=1}^{3}\left\{\frac{A(3\pi^2-4)}{12}\left[(a_{2i}-a^0_{2i}) + (a'_{2i}-a'^0_{2i})\right] + v^0_{5i}\right\}\frac{\ln(n+1/2)}{(n+1/2)^2} + \frac{\pi}{2}\Big\{M_\lambda + M_\mu + (\lambda^0_0+\mu^0_0) - \\
& - (\lambda_0+\mu_0) + \sum_{i=1}^{3}\Big\{\frac{1}{2}\left((a_{2i}+a'_{2i}) + (a^0_{2i}+a'^0_{2i})\right) + 4(a_{2i}-a^0_{2i}) + \left(3 + \frac{2A\ln(3/2)}{3\pi}\right) \times \\
& \times\left[(a_{2i}-a^0_{2i}) + (a'_{2i}-a'^0_{2i})\right] - \frac{\pi^2}{3}\left[(a_{2i}-a^0_{2i})^2 + (a'_{2i}-a'^0_{2i})^2\right] + \frac{3^{p_i-1}}{2^{p_i-1}}\left[(a_{5i}-a^0_{5i}) + (a'_{5i}-a'^0_{5i})\right] + \\
& + \frac{1}{2^{2p_i-2}3^{3-2p_i}}\left[(\gamma_{1i}-\gamma^0_{1i}) + (\gamma'_{1i}-\gamma'^0_{1i})\right] + \frac{\ln(3/2)}{2^{p_i-1}3^{2-p_i}}\left[(\gamma_{2i}-\gamma^0_{2i}) + (\gamma'_{2i}-\gamma'^0_{2i})\right] + \\
& + \frac{1}{2^{p_i-1}3^{2-p_i}}\left[(\gamma_{3i}-\gamma^0_{3i}) + (\gamma'_{3i}-\gamma'^0_{3i})\right] + \frac{a_{10i}-a'_{10i}}{a_{2i}-a'_{2i}}\frac{a^0_{10i}-a'^0_{10i}}{a^0_{2i}-a'^0_{2i}} + v^0_{6i}\Big\}\Big\}\frac{1}{(n+1/2)^2} + \\
& + O\left(\frac{\ln^2 n}{n^{4-p_3}}\right). \qquad (2.18)
\end{aligned}
$$

Remark 1. The asymptotic formula (2.18) for the α_n holds from the same sufficiently large n onwards. For small n the numbers α_n must be calculated directly from the formula (1.18), where the difference $H_2 - H_1$ is given by equation (2.3).

Remark 2. If we specify further terms in the asymptotic formulas (2.1) and (2.2) for the α_n and μ_n, then this method can be used to calculate further terms in the asymptotic formula for the α_n, but we small not give these here because of the heavy computations.

3. The Inverse Sturm-Liouville Problem.

In this section we consider the inverse Sturm-Liouville problem.

Theorem. *Suppose that we are given two sequence of numbers* $\{\lambda_n\}_{n=0}^{\infty}$ *and* $\{\mu_n\}_{n=0}^{\infty}$, $(n=0,1,\ldots)$. *Satisfying the following conditions:*

1. λ_n *and* μ_n *are both ordered, that is* $\lambda_0<\mu_0<\lambda_1<\mu_1\ldots$
2. $\lambda_n=(n+1/2)^2+\sum_{i=0}^{2}a_{1i}(n+1/2)^{p_{3-i}-1}+a_3(n+1/2)^{2p_3-3}+\frac{A}{\pi}\ln(n+1/2)+$

$$+a_8(n+1/2)^{3p_3-5}+a_{11}(n+1/2)^{4p_3-7}+a_{14}(n+1/2)^{5p_3-9}+$$

$$+\sum_{i=0}^{2}a_{2i}+a_{43}\frac{\ln(n+1/2)}{(n+1/2)^{2-p_3}}+a_{53}\frac{1}{(n+1/2)^{2-p_3}}+$$

$$+a_{32}\frac{1}{(n+1/2)^{3-2p_2}}+a_{93}\frac{\ln(n+1/2)}{(n+1/2)^{4-2p_3}}+\gamma_{13}\frac{1}{(n+1/2)^{4-2p_3}}+a_{123}\frac{\ln(n+1/2)}{(n+1/2)^{6-3p_3}}+$$

$$+\gamma_{73}\frac{1}{(n+1/2)^{6-3p_3}}+a_{31}\frac{1}{(n+1/2)^{3-2p_1}}+a_{41}\frac{\ln(n+1/2)}{(n+1/2)^{2-p_1}}+a_{51}\frac{1}{(n+1/2)^{2-p_1}}+$$

$$+a_{82}\frac{1}{(n+1/2)^{5-3p_2}}+a_{42}\frac{\ln(n+1/2)}{(n+1/2)^{2-p_2}}+a_{52}\frac{1}{(n+1/2)^{2-p_2}}+a_{1n}$$

$$\mu_n=(n+1/2)^2+\sum_{i=0}^{2}a'_{1i}(n+1/2)^{p_{3-i}-1}+a'_3(n+1/2)^{2p_3-3}+\frac{A}{\pi}\ln(n+1/2)+$$

$$+a'_8(n+1/2)^{3p_3-5}+a'_{11}(n+1/2)^{4p_3-7}+a'_{14}(n+1/2)^{5p_3-9}+\sum_{i=0}^{2}a'_{2i}+a'_{43}\frac{\ln(n+1/2)}{(n+1/2)^{2-p_3}}+$$

$$+a'_{53}\frac{1}{(n+1/2)^{2-p_3}}+a'_{32}\frac{1}{(n+1/2)^{3-2p_2}}+a'_{93}\frac{\ln(n+1/2)}{(n+1/2)^{4-2p_3}}+\gamma'_{13}\frac{1}{(n+1/2)^{4-2p_3}}+$$

$$+a'_{123}\frac{\ln(n+1/2)}{(n+1/2)^{6-3p_3}}+\gamma'_{73}\frac{1}{(n+1/2)^{6-3p_3}}+a'_{31}\frac{1}{(n+1/2)^{3-2p_1}}+a'_{41}\frac{\ln(n+1/2)}{(n+1/2)^{2-p_1}}+$$

$$+a'_{51}\frac{1}{(n+1/2)^{2-p_1}}+a'_{82}\frac{1}{(n+1/2)^{5-3p_2}}+a'_{42}\frac{\ln(n+1/2)}{(n+1/2)^{2-p_2}}+a'_{52}\frac{1}{(n+1/2)^{2-p_2}}+a'_{1n}$$

asymptotic formulas are valid. Here $a_{2i}\neq a'_{2i},(i=1,2,3)$ *and* $\sum a_{1n}^2,\sum a_{1n}'^2$, *are both convergent series.*

Then there exist $q(x)\in C[a,b]$ *and* H_1,H_2 *real numbers so that* $\{\lambda_n\}$ *is a spectrum of (1.1)-(1.2) and* $\{\mu_n\}$ *is a spectrum of (1.1)-(1.3) and* $H_1-H_2=\pi(a'_{2i}-a_{2i})$.

Proof. Suppose that we are given sequence of numbers $\{\lambda_n\}$ and $\{\mu_n\}$ with the above properties. We construct the sequence of α_n by means of the formula (1.18). It satisfies the asymptotic estimate (2.18). We shall show that all $\alpha_n>0$. The first condition of the theorem shows, that λ_n and μ_n are interlaced. Therefore, for $n=0,1,\ldots$ $\mu_n>\lambda_n$ or $\mu_n<\lambda_n$. α_n's are given by formula (1.18) or

$$\alpha_n=\alpha_n^0\frac{\lambda_n-\mu_n}{\lambda_n^0-\mu_n^0}\frac{\lambda_n-\lambda_0}{\lambda_n^0-\lambda_k^0}\frac{\lambda_n-\mu_0}{\lambda_n^0-\mu_0^0}\prod_{k=1}^{\infty}{}'\left(\frac{\lambda_n-\lambda_k}{\lambda_n^0-\lambda_k^0}\right)\prod_{k=1}^{\infty}{}'\left(\frac{\lambda_n-\mu_k}{\lambda_n^0-\mu_k^0}\right).$$

Here, we can write $\prod_{k=1}^{\infty}{}'\frac{\lambda_n-\lambda_k}{\lambda_n^0-\lambda_k^0}>0$, $\prod_{k=1}^{\infty}{}'\frac{\lambda_n-\mu_k}{\lambda_n^0-\mu_k^0}>0$.

Since $\alpha_n^0 > 0$ and λ_n, μ_n and λ_n^0, μ_n^0 are interlaced. It is obvious from α_n formula that $n = 0,1,...,\alpha_n > 0$. Since Sturm-Liouville operators can be built according to $\{\lambda_n\}$ and $\{\mu_n\}$ sequences. As a result, we have proved that for given $\{\lambda_n\}$ and $\{\mu_n\}$ sequences Sturm-Liouville operators can be built [3].

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