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KAZIMOV T.G.

ON ONE BOUNDARY VALUE PROBLEM FOR MODEL EQUATION OF A MIXED TYPE

Abstract

The model equation of a mixed type is considered. For the first time the boundary condition when the linear combination of values to the solution on one characteristics with values of derivative to the solutions on other characteristics with variable coefficients is used. The uniqueness of the solution to the problem is based on the proved principle of extremum. Existence of the solution is proved by the method of singular integral equation.

Let D be a finite simply connected domain on the xy plane, bounded by Jordan line σ with the ends at points $A(0,0)$, $B(1,0)$ arranged in upper half-plane $y>0$ and with characteristics $AC{:}\,x+y=0, BC{:}\,x-y=1$ outgoing from the point $C(1/2,-1/2)$ of equation

$$u_{xx}+\operatorname{sgn} y\, u_{yy}=0. \tag{1}$$

Denote by D_1 and D_2, an elliptic and a hyperbolic parts of the mixed domain D. $J \equiv AB$ is a unit interval.

Formulation of the problem T_6. It is required to find the function $u(x,y)$ with the following properties:

1) $u(x,y)$ is a solution of equation (1) in domain $D \setminus J$;
2) $u(x,y) \in C(\overline{D}) \cap C'(D) \cap C^2(D \setminus J)$ and u_x, u_y in proximity to points A and B car turn into infinity of the order less than unit;
3) $u(x,y)$ satisfies the boundary conditions

$$u(x,y)\big|_{\sigma}=\varphi(x,y), \tag{2}$$

$$u\left(\frac{x}{2},-\frac{x}{2}\right)+\beta(x)\frac{d}{dx}u\left(\frac{x+1}{2},\frac{x-1}{2}\right)=\delta(x), \quad \forall x \in J, \tag{3}$$

where $\varphi(x,y)$, $\beta(x)$ and $\delta(x)$ are given continuous functions and $\beta(x), \delta(x) \in H(\overline{J}) \cap$ $\cap C^2(J)$.

In case $\beta(x) \equiv 0$ the problem T_6 coincides with the known problem of Lavrentyev- Bitsadze [1]. It should be noted, that condition (3) is considered for the first time.

Solution to the problem T_6. With the help of Dalamber's formula, it is not difficult to show that any solution to the problem T_6, if it exists, taking into account condition (3) satisfies the following correlation on degeneration line AB

$$\tau'(x)+p(x)\tau(x)=q(x), \quad \forall x \in J, \tag{4}$$

where $\tau(x) = u(x,0)$, $\gamma(x) = u_y(x,0)$, $p(x) = 1/\beta(x)$, $q(x) = p(x)\int_0^x \gamma(t)dt - \gamma(x) + f_0(x)$,

$f_0(x) = (2\delta(x) - \varphi(0))/\beta(x)$, $\beta(x) \neq 0$.

Formulate the known principle of extremum for the problem T_6.

Let $\delta(x) \equiv 0, \varphi(0) = 0$ and $\beta(x) < 0$ for $\forall x \in J$, then the positive maximum and the negative minimum of the solution $u(x,y)$ to the problem T_6, in closed domain $\overline{D}$, are obtained on σ.

In fact, if suppose that $\max\limits_{\overline{D}_1} u(x,y) = \tau(\xi) > 0$, then it is obvious that $\xi \,\overline{\in}\, D_1$. Let $\xi \in J$. In this case from (4) for $\gamma(x)$ we shall have an expression

$$\gamma(x) = -2p(x)V^{-1}(x)\int_0^x p(t)V(t)\tau(t)dt - 2p(x)\tau(x) - \tau'(x), \tag{5}$$

where $V(x) = \exp\left(-\int_0^x p(s)ds\right)$.

Taking the value expression (5) at point ξ, taking into account that $\tau'(\xi) = 0$, $\beta(\xi) < 0$ and after simple transformations we will obtain:

$$\gamma(\xi) = 2p(\xi)V^{-1}(\xi)\int_0^\xi p(t)V(t)[\tau(\xi) - \tau(t)]dt - 2p(\xi)V^{-1}(\xi)\tau(\xi) > 0 .$$

The last contradicts to the principle of Zarembo-Zhiro. Consequently, $\xi \in \sigma$. The same conclusion would be made for the minimum. From this principle it follows the uniqueness of solution to the problem T_6.

Existence of solution to the T_6 is reduced to finding the function $\gamma(x)$ [1]. Without limitation of generality, we can suppose $\varphi = 0$.

In addition, we will suppose that σ coincides with half circumference σ_0 with ends at points A and B. In this case the correlation between $\tau(x)$ and $\gamma(x)$ on J from elliptic part D_1 of the mixed domain D has the form [1]

$$\tau'(x) + \frac{1}{\pi}\int_0^1\left(\frac{1}{t-x} + \frac{1-2t}{t+x-2tx}\right)\gamma(t)dt = 0. \tag{6}$$

Having solved the equation (4) with respect to $\tau(x)$, for $\tau'(x)$ we will get the expression

$$\tau'(x) = 2p(x)V^{-1}(x)\int_0^x V(t)\gamma(t)dt - \gamma(x) + f(x), \tag{7}$$

where $f(x) = f_0(x) - p(x)V^{-1}(x)\int_0^x V(t)f_0(t)dt$.

Excluding $\tau'(x)$ from (6) and (7), after easy transformations we will get the following singular integral equation

$$\gamma(x) - \frac{1}{\pi}\int_0^1\left(\frac{1}{t-x} + \frac{1-2t}{t+x-2tx}\right)\gamma(t)dt + \int_0^x g(x,t)\gamma(t)dt = f(x), \tag{8}$$

where $g(x,t) = -2p(x)V^{-1}(x)V(t)$.

Taking into account the identity

$$\frac{1}{t-x}+\frac{1-2t}{t+x-2tx}=\frac{2t(1-t)}{t^2\left(1-2x+2x^2\right)-x^2\left(1-2t+2t^2\right)},$$

it is easy to check that change of variables

$$\tau=\frac{t^2}{1-2t+2t^2},\quad y=\frac{x^2}{1-2x+2x^2}$$

reduces (8) to the equation of Garleman type

$$\rho(y)-\frac{1}{\pi}\int_0^1\frac{\rho(\tau)d\tau}{\tau-y}=R(y), \tag{9}$$

where

$$\rho(y)=\left(1-2x+2x^2\right)\gamma(x),\quad R(y)=\left(1-2x+2x^2\right)\left[f(x)-\int_0^x g(x,t)\gamma(t)dt\right],$$

$$x=\frac{\sqrt{y}}{\sqrt{y}+\sqrt{1-y}}.$$

Equation (9) by the well-known regularization method of Garleman-Vecua [2] we can reduce to the equivalent integral of Fredholm equation of the second type, whose absolute solvability follows from uniqueness of solution to the problem T_6.

References

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Kazimov T.G.
Informational Telecommunications Scientific Center of AS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-90-16.

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