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APPROXIMATED SOLUTION OF THE EQUATION OF A SECOND KIND WITH A SUM OF TWO OPERATORS

Abstract

In this work the problem about an approximated solution of the equation of the second kind with a sum of two operators (monotone and contracting) in Banach space is considered. The theorems of a univalent resolvability, about an evaluation of a velocity of convergence of a sequence of approximations to a solution of the equation are proved. Such generalized outcome of Iu.L. Gaponenco about the approximated solution of the operator equation of second kind with a monotone operator is shown.

Key words: *a monotonicity, an approximated solution, operator.*

The second kind equation with Lipschitzian-continuous and monotone operator was studied in Yu.L.Gaponenco's paper by the continuation in parameter method. In this paper the author generalizes the previous result for the second kind equation with a sum of two operators (monotone and contractive). Theorems on a univalent solvability of the equation, on the estimation of convergence velocity of a succession of approximate solutions to the solution of the inverse operator norm are proved.

Let A and B be non-linear operators acting on the Banach space X. Consider the equation

$$x + A(x) + B(x) = f, \tag{1}$$

where f is a given element from X.

Definition ([1]). *The operator A is called monotone in X, if for any $x, y \in X$ and any $\lambda > 0$ the inequality*

$$\|x - y + \lambda[A(x) - A(y)]\| \geq \|x - y\|$$

holds.

Approximate solution of the equation having the form $x + A(x) = f$ with a monotone operator A acting on a Banach space or in a metrizable linear topological space was studied in [1], [2]. We use the result from [1] to study the equation (1). The solution of an operator equation whose left-hand side is represented in the form of the sum of two operators was studied by many authors, for instance in [3-5].

Theorem 1. *Let A and B be operators acting on a Banach space X and the following conditions be fulfilled:*

1) A is monotonic and it satisfies the Lipschitz equation:

2) B is a contractive operator.

Then equation (1) has a unique solution.

Proof. Assume

$$C(x) \equiv x + A(x) = y. \tag{2}$$

As is known, (see [1]) by fulfilling condition 1) the equation $C(x) = y$ has a unique solution for any given y. Therefore, there exists the operator C^{-1} acting on all space X. Prove that C^{-1} satisfies the Lipschitz equation with constant $L = 1$.

Indeed, let $y_1, y_2 \in X$, assume

$$C^{-1}(y_1)=x_1, \quad C^{-1}(y_2)=x_2.$$

By monotonicity of the operator A we have

$$\|C^{-1}(y_1)-C^{-1}(y_2)\|=\|x_1-x_2\|\le\|x_1-x_2+A(x_1)-A(x_2)\|=$$
$$\|C(x_1)-C(x_2)\|=|y_1-y_2|$$

Thus

$$\|C^{-1}(y_1)-C^{-1}(y_2)\|\le\|y_1-y_2\|.$$

We have

$$x+A(x)+B(x)\equiv C(x)+B(x)=f,$$

or

$$y+BC^{-1}(y)=f. \qquad (3)$$

Let a contraction coefficient of the operator B be $q(0\le q<1)$, then $\|BC^{-1}(y_1)-BC^{-1}(y_2)\|\le q\ \|C^{-1}(y_1)-C^{-1}(y_2)\|\le q\|y_1-y_2\|$.

Equation (3) has a unique solution y by principle of a contracted mapping. By putting y in (2) we have $x+A(x)=y$. As it was proved above, this equation has a unique solution x. It is the unique solution of the equation (1). The theorem has been proved.

Now consider an approximate solution problem of the equation

$$x+A(x)+B(x)=f.$$

Let all conditions of Theorem 1 be fulfilled. Then there exists a unique solution of the equation (1). In order to find this solution, we have to solve equations (2) and (3).

Let L be a Lipschitz constant of the operator A, and N be a positive integer, such that $(L/N)<1$. Assume $\varepsilon=1/N$. Then, equation (1) has the form

$$x+N\varepsilon A(x)+B(x)=f. \qquad (4)$$

Introduce the substitutions

$$\begin{aligned} &C_1(x)\equiv x+\varepsilon A(x)=u^{(1)}\\ &C_2(u^{(1)})\equiv u^{(1)}+\varepsilon AC_1^{-1}(u^{(1)})=u^{(2)}\\ &\dots\\ &C_N(u^{(N-1)})\equiv u^{(N-1)}+\varepsilon AC_1^{-1}C_2^{-1}\dots C_{N-1}^{-1}(u^{(N-1)})=y \end{aligned} \qquad (5)$$

After the substitution, the equation (4) has the form

$$y+BC_1^{-1}\dots C_N^{-1}(y)=f. \qquad (6)$$

Note that with these substitutions the operator has the representation

$$C(x)=C_NC_{N-1}\dots C_1(x), x\in X.$$

By monotonicity of the operator A we can prove that the operators $C_i^{-1}(i=1,2,\dots N)$ exist, they are defined on all X and they satisfy the Lipschitz condition with constant that equals to one (see [1]). Therefore $\varepsilon A, \varepsilon AC_1^{-1}\dots C_i^{-1}$ and $BC_1^{-1}\dots C_i^{-1}(i=1,2,\dots N)$ are contracting operators. Hence, each equation in (5) and (6) has a unique solution and it may be solved by an approximation method of a simple iteration. First find y and then $u^{(N-1)}, u^{(N-2)},\dots,u^{(1)},x$. X is the desired solution of the equation (1). Approximate solution process of equations is represented as:

$$x_{n+1} = -\varepsilon A(x_n) + u_p^{(1)}, n = 0,1,2...$$
$$u_{n+1}^{(1)} = -\varepsilon AC_1^{-1}\left(u_n^{(1)}\right) + u_p^{(2)}, n = 0,1,2...$$
$$u_{n+1}^{(2)} = -\varepsilon AC_1^{-1}C_2^{-1}\left(u_n^{(2)}\right) + u_p^{(3)}, n = 0,1,2,...$$
$$...$$
$$u_{n+1}^{(N-1)} = -\varepsilon AC_1^{-1}C_2^{-1}...C_{N-1}^{-1}\left(u_n^{(N-1)}\right) + y_p, n = 0,1,2,... \quad (7)$$
$$y_{n+1} = -BC_1^{-1}C_2^{-1}...C_N^{-1}(y_n) + f, n = 0,1,2,..., p = 0,1,2,...$$

where initial approximations $x_0, u_0^{(1)}, u_0^{(2)}, \dots, u_0^{(N-1)}, y_0$ are arbitrary from X. Now estimate the velocity of the convergence of x_n to the solution x of equation (1). We are to solve equations (2) and (3). For simplicity assumes that $A(0) = 0$ and the number of steps at each iterational process is the same and equals to s.

Indeed, if $A(0) \neq 0$, we put $T(x) = A(x) - A(0)$. Then $T(0) = 0$, $T(x) - T(y) = A(x) - A(y)$ and the equation $x + T(x) = f - A(0)$ is equivalent to the equation (1). Besides, the operator T is monotone and it also satisfies the Lipschitz condition with a constant that is equal to L. Finally substitute $f - A(0)$ instead of f.

The solution of the equation (3) is the limit of the sequence $\{y_n\}$, where

$$y_{n+1} = -BC^{-1}(y_n) + f, \quad n = 0,1,2,\dots, y_0 \in X. \quad (8)$$

In order to find $C^{-1}(y_n)$ we have to solve the equation

$$x + A(x) = y_n. \quad (9)$$

Let $x^{(n)}$ and $x_k^{(n)}$ $(k = 0,1,2,\dots)$ respectively be the exact and approximate solutions of the equation (9). As is known from [1] we have

$$\left\|x_s^{(n)} - x^{(n)}\right\| \le \theta^{s+1} \frac{[\exp(L) - 1]\|y_n\|}{(1-\theta)[\exp(\theta) - 1]}, \quad (10)$$

where $\theta = (L/N) < 1, \quad s = 0,1,2,\dots$.

We have from (8)

$$\|y_n - y_{n-1}\| \le \left\|BC^{-1}y_{n-1} - BC^{-1}y_{n-2}\right\| \le q\|y_{n-1} - y_{n-2}\| \le \dots \le q^{n-1}\|y_1 - y_0\|,$$
$$\|y_n\| \le \|y_0\| + \|y_n - y_0\| \le \|y_n - y_{n-1}\| + \|y_{n-1} - y_{n-2}\| + \dots + \|y_1 - y_0\| + \|y_0\|$$
$$\le \left(\sum_{k=0}^{n-1} q^k\right)\|y_1 - y_0\| + \|y_0\| = \frac{1-q^n}{1-q}\|y_1 - y_0\| + \|y_0\| < (1/1-q)\|y_1 - y_0\| + \|y_0\|.$$

If we take $y_0 = 0$ and consider that $C^{-1}(0) = 0$, then $\|y_1 - y_0\| = \left\|-BC^{-1}(0) + f\right\| = \|f - B(0)\|$. Hence for $n = s$ we have $\|y_s\| < (1/1-q)\|f - B(0)\|$. We have from (10)

$$\left\|x_s^{(s)} - x^{(s)}\right\| \le \theta^{s+1} \frac{[\exp(L) - 1]}{(1-\theta)[\exp(\theta) - 1]} \frac{1}{(1-q)}\|f - B(0)\|. \quad (11)$$

We denote by Δ_s the right-hand side of the inequality (11). We get error by calculating $C^{-1}y_s$ is Δ_s. On the other hand, B is contractive operator. Therefore, the error by calculating $BC^{-1}y_s$ equals to $q\Delta_s$. The error at an approximate solution of the equation

(3) is equal to the sum of calculation error and the error of the iteration process. We have (see [3])

$$\|y_s - y\| \le \frac{q^s}{1-q}\|y_1 - y_0\| + q\Delta_s = \frac{q^s}{1-q}\|f - B(0)\| + q\Delta_s .$$

Let x_s be an approximate solution of the equation (1), considering that the operator C^{-1} satisfies the Lipschitz equation with a coefficient that equals to one. Therefore the error $\|x_s - x\|$ equals to the sum of error at the right-hand side of the equation (2) and the error of the iterational process. And we have

$$\|x_s - x\| \le \Delta_s + \frac{q^s}{(1-q)}\|f - B(0)\| + q\Delta_s ,$$

or

$$\|x_s - x\| \le \theta^{s+1}\frac{[\exp(L)-1]}{(1-\theta)[\exp(\theta)-1]}\frac{\|f - B(0)\|}{(1-q)}(q+1) + \frac{q^s}{1-q}\|f - B(0)\| , \qquad (12)$$

where $\theta = (L/N) < 1, \quad s = 0,1,2,...$

Note that x_s generally depends on N, therefore we denote $x(N,s) \equiv x_s$. Thus, we proved the following theorem:

Theorem 2. *Let operators A and B satisfy all conditions of Theorem 1, and $A(0) = 0$. The approximate solution $x(N,s)$ of the equation (1) is obtained by iteration method. Convergence velocity of the succession of approximate solutions are expressed by formula*

$$\|x(N,s) - x\| \le \theta^{s+1}\frac{[\exp(L)-1]\|f - B(0)\|}{(1-\theta)[\exp(\theta)-1](1-q)}(q+1) + \frac{q^s}{1-q}\|f - B(0)\| ,$$

where $\theta = (L/N) < 1, \; s = 0,1,2,...$

Note, that if the operator $B \equiv 0$, the equation (1) has the form of $x + A(x) = f$ and the estimate (12) has the form $\|x_s - x\| \le \Delta_s$, known in [1]. When $A \equiv 0$, the equation (1) has the form $x + B(x) = f$ we have a theorem on a fixed point of a contractive operator.

Theorem 3. *Let operators A and B satisfy the conditions of theorem 1. Then the inverse operator F of the operator $I + A + B$ satisfies the Lipschitz condition:*

$$\|F(u) - F(v)\| \le \left(\frac{1}{1-q}\right)\|u - v\| \quad u,v \in X ,$$

where q is a contraction coefficient of the operator B.

Proof. By theorem 1, the equation $x + A_x + B_x = f$ has a unique solution at any given f. Therefore the operator $I + A + B$ has its inverse one. We denote it by F. The operator F is defined on all space X.

Let $u,v \in X$. Set $F(u) = x, \; f(v) = x_2$. Then $u = F^{-1}(x_1), \; v = F^{-1}(x_2)$.

$$\|u - v\| = \|F^{-1}(x_1) - F^{-1}(x_2)\| = \|(x_1 - x_2) + A(x_1) - A(x_2) + B(x_1) - B(x_2)\| \ge$$

$$\|(x_1 - x_2) + A(x_1) - A(x_2)\| - \|B(x_1) - B(x_2)\| \ge \|x_1 - x_2\| - q\|x_1 - x_2\| = (1-q)\|x_1 - x_2\|$$

Thus we obtain

$$\|F(u - F(v))\| \le (1/1-q)\|u - v\| .$$

Theorem has been proved. We have from this theorem

Corollary. *Let f_1, f_2 be the given elements of the right-hand side of the equation (1), and x_1, x_2 are their corresponding solutions. Then*

$$\|x_1 - x_2\| \leq \left(\frac{1}{1-q}\right)\|f_1 - f_2\|.$$

References

[1]. Гапоненко Ю.Л. *Метод продолжения по параметру для уравнения II рода с Липшиц-непрерывным и монотонным оператором.* Журн. В. М. и М. Ф., 1986, №8, с. 1123-1131.

[2]. Khuat V.N. *Iteration method for solution of operator equation in metrizable topological vector space.* J. Math., 1990, v.18, №2, p.13-22.

[3]. Качуровский Р.И. *Нелинейные монотонные операторы в банаховых пространствах.* У.М.Н., 1968, т.23, №2, с.121-168.

[4]. Красносельский М.А., Липшиц Е.А, Соболев А.В. *Позитивные линейные системы.* М.; Наука, 1985.

[5]. Кхуат В.Н., Мамедов Я.Д. *Сходимости двусторонних приближений к решению операторных уравнений с немонотонной правой частью.* «Вестник Бакинского ун-та», №1, 1992, с. 95-99.

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