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VARIETIES WITH DEFINABLE PRINCIPAL CONGRUENCES: HOW CONGRUENCES IN A (SUB)DIRECT PRODUCT ARE DETERMINED BY FACTORS

To Professor L.N. Shevrin, on the occasion of his 65[th] birthday.

Abstract

The key result is a characterization of varieties with definable principal congruences (and with CEP) by some conditions on the congruences of (sub)direct products. In passing we give a generalization of Magari's idea of ideal congruences for (sub)direct products of algebras in a variety. AMS Subject Classification: 08B99.

1. Introduction

A variety $\mathbf{K}$ of algebras has definable principal congruences (DPC) if there is a first-order formula in the language of $\mathbf{K}$ that defines all principal congruences for the algebras in $\mathbf{K}$. This concept was introduced by J.Baldwin and J. Berman [1] in the study of cardinality estimates for subdirectly irreducible algebras in a variety. There are a number of results characterizing DPC in different kinds of algebras, a visual example is due to K.Baker [2]: a locally finite variety of groups has DPC iff it is satisfies the commutator identity $[x,y,x]$. J. Baldwin and J. Berman [3] considered notions both weaker and stronger than DPC (for arbitrary classes of algebras). In particular (see [3], p.259) if there is only one disjunct (or congruence scheme) in the (equivalent form of the) defining formula then DPC is reduced to E. Fried, G. Gratzer, and R. Quackenbush's notion of uniform congruence scheme [4]. In [5] we considered the general case: a finite number of disjuncts (or congruence schemes); it turned out that for DPC- varieties there is a likeness with the situation around the uniform (restricted) congruence scheme. Also, we had seen how principal congruences in a (sub)direct product are determined by the factors. Now our aim is to extend this to arbitrary congruences.

For terminology we shall generally follow G. Gratzer [6] with the exception that we shall refer to his algebraic functions as polynomial functions and to his polynomials as terms.

2. Preliminaries

This section presents some definitions and results that are needed later in the paper. For the sake of conveniences we also repeat basic definitions and results of [5].

Mal'cev's Lemma (see [6]) gives a description of principal congruences in universal algebras and the general scheme contains many parameters. E. Fried, G. Gratzer, and R. Quackenbush [4] defined a uniform congruence scheme (see details below): a variety $\mathbf{K}$ has a uniform congruence scheme if in the whole class $\mathbf{K}$ Mal'cev's Lemma is applied with the same permament parameters.

Definition 2.1 ([4]). *A <u>congruence scheme</u> S for a given type τ is given by $m+1$- ary $(m\geq 0)$ terms $t_0,\dots,t_{k-1}$ $(k\geq 1)$ and by a map $f:\{0,\dots,k-1\}\to\{0,1\}$. Let $\mathcal{A}$ be an algebra whose type includes τ and let $a_0,a_1,b_0,b_1\in A$. Scheme S is satisfied in $\mathcal{A}$ for a_0,a_1,b_0,b_1 (or $S(a_0,a_1,b_0,b_1)$ holds in $\mathcal{A}$; in notation $\mathcal{A}\models S(a_0,a_1,b_0,b_1)$) iff there exist $c_1,\dots,c_m\in A$ satisfying*

$$b_0=t_0(a_{f(0)},c_1,\dots,c_m),$$

$$t_i\left(a_{1-f(i)}, c_1, \ldots, c_m\right) = t_{i+1}\left(a_{f(i+1)}, c_1, \ldots, c_m\right), \quad i = 1,\ldots,k-2,$$
$$t_{k-1}\left(a_{1-f(k-1)}, c_1, \ldots, c_m\right) = b_1.$$

Obviously, by Mal'cev's Lemma, $c \equiv d(\theta(a,b))$ in an algebra $\mathcal{A}$ iff there exists a congruence scheme S such that $\mathcal{A} \models S(a_0, a_1, b_0, b_1)$.

Definition 2.2. *A variety* $\mathbf{K}$ *has* n *congruence schemes (nCS),* $n \geq 1$, *iff there exist congruence schemes* $S_1,\ldots,S_n$ *such that for any algebra* $\mathcal{A} \in \mathbf{K}$ *and any elements* a, b, c, d *in* $\mathcal{A}$ *there is a congruence scheme* $S_i \in \{S_1,\ldots,S_n\}$ *such that*

$$c \equiv d(\theta(a,b)) \Leftrightarrow \mathcal{A} \models S_i(a, b, c, d).$$

As noted in [4], p. 257, if a variety has DPC then, in fact, it has nCS for some n.

Definition 2.3 ([5]). *A variety* $\mathbf{K}$ *has* *n equationally definable principal congruences (nEDPC) iff there are* n *sets of equations*

$$\left\{p_j^1 = q_j^1 \middle| j \in J_1\right\}, \ldots, \left\{p_j^n = q_j^n \middle| j \in J_n\right\}$$

such that for any algebra $\mathcal{A} \in \mathbf{K}$ *and any* $a, b, c, d \in A$, *there is* $J_i \in \{J_1,\ldots,J_n\}$ *such that* $c \equiv d(\theta(a,b))$ *is equivalent to the existence of* $e_0, e_1, \ldots \in A$ *such that*

$$\left\{p_j^i(a,b,c,d,e_0,e_1,\ldots) = q_j^i(a,b,c,d,e_0,e_1,\ldots) \middle| j \in J_i\right..$$

Definition 2.4. *A variety* $\mathbf{K}$ *has* *n factor determined principal congruences (nFDPC) on direct products* *iff whenever* $\mathcal{A}_j \in \mathbf{K}$ *for* $j \in J$, $a_j, b_j, c_j, d_j \in A_j$ *and* $c_j \equiv d_j\left(\theta\left(a_j, b_j\right)\right)$ *for all* $j \in J$, *then there are subsets* $J_1,\ldots,J_n \subseteq J$ *such that the following conditions hold:*

(i) $J_1 \cup \cdots \cup J_n = J$, *and*

(ii) for any $J_i \in \{J_1, \ldots, J_n\}$ *there holds* $c_{J_i} \equiv d_{J_i}\left(\theta\left(a_{J_i}, b_{J_i}\right)\right)$ *in the algebra* $\mathcal{A}_{J_i} = \Pi\left(\mathcal{A}_j \middle| j \in J_i\right)$, *where* $x_{J_i} = \langle x_j | j \in J_i \rangle$ *for* $x \in \{a,b,c,d\}$.

Definition 2.5. *A variety* $\mathbf{K}$ *has* *n algebras with universal principal congruences (nAUPC) iff there exist algebras* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *in* $\mathbf{K}$ *and elements* $a_1, b_1, c_1, d_1 \in A_1, \ldots, a_n, b_n, c_n, d_n \in A_n$ *such that*

$$c_1 \equiv d_1\left(\theta^{\mathcal{A}_1}(a_1, b_1)\right), \ldots, c_n \equiv d_n\left(\theta^{\mathcal{A}_n}(a_n, b_n)\right)$$

and for any algebra $\mathcal{B} \in \mathbf{K}$ *and elements* $a', b', c', d' \in B$ *if* $c' \equiv d'\left(\theta^{\mathcal{B}}(a', b')\right)$ *then there exists an algebra* $\mathcal{A}_i \in \{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ *and a homomorphism* φ *of* $\mathcal{A}_i$ *into* $\mathcal{B}$ *satisfying* $a' = \varphi(a_i)$, $b' = \varphi(b_i)$, $c' = \varphi(c_i)$, *and* $d' = \varphi(d_i)$.

Theorem 2.6 ([5]). *Let* $\mathbf{K}$ *be a variety. There is* $n \geq 1$ *such that t.f.a.e.: (i)* $\mathbf{K}$ *has DPC; (ii)* $\mathbf{K}$ *has nCS; (iii)* $\mathbf{K}$ *has nEDPC; (iv)* $\mathbf{K}$ *has nFDPC on direct products; (v)* $\mathbf{K}$ *has nAUPC.*

We shall also consider relationship of DPC and CEP (congruence extension property). However, first at all we remind of the definition of CEP: a variety $\mathbf{K}$ is said to have CEP iff for any algebra $\mathcal{A} \in \mathbf{K}$, subalgebra $\mathcal{B}$ of $\mathcal{A}$, and congruence θ of $\mathcal{B}$, there exists a congruence Φ of $\mathcal{A}$ whose restriction to $\mathcal{B}$ is θ.

Definition 2.7. *n restricted congruence schemes (nRCS) is a nCS (as in 2.2) with the requirement that in all congruence schemes* $m = 4$ *and* $c_1 = a$, $c_2 = b$, $c_3 = c$, *and* $c_4 = d$. *If in 2.3 all* $p_j^1, q_j^1, \ldots, p_j^n, q_j^n$ *are 4-ary, then* $\mathbf{K}$ *is said to have nREDPC (R means restricted). nRAUPC stands for nAUPC with the additional requirement that* $\mathcal{A}_i$ *is generated by* a_i, b_i, c_i, *and* d_i *for all* $i = 1,\ldots,n$.

Definition 2.8. *A variety* $\mathbf{K}$ *has nFDPC* *on subdirect products* *iff whenever* $\mathcal{A}_j \in \mathbf{K}$ *for* $j \in J$, $a_j, b_j, c_j, d_j \in A_j$, $c_j \equiv d_j(\theta(a_j, b_j))$ *for all* $j \in J$, *and* $\mathcal{A} \le \Pi(\mathcal{A}_j | j \in J)$ *is a subdirect representation then there are subsets* $J_1, \dots, J_n \subseteq J$ *such that the following conditions hold:*

(i) $J_1 \cup \dots \cup J_n = J$, *and*

(ii) for every $J_i \in \{J_1, \dots, J_n\}$ *there holds* $c_{J_i} \equiv d_{J_i}(\theta(a_{J_i}, b_{J_i}))$ *in the subalgebra* $\mathcal{A}^*_{J_i} \le \mathcal{A}_{J_i} = \Pi(\mathcal{A}_j | j \in J_i)$ *such that* $\mathcal{A}^*_{J_i}$ *is the projection of* $\mathcal{A}$ *to the all coordinates in* J_i.

Definition 2.9. *Let* Σ *be a join-semilattice. An ordered pair* (a,b) *in* $\Sigma \times \Sigma$ *is said to be* *n dually Brouwerian* *(nDB) iff there exist* $c_1, \dots, c_n$ *in* Σ *such that for any* $t \in \Sigma$

$$a \vee t \ge b \Leftrightarrow t \ge c_i \quad \text{for some} \quad c_i \in \{c_1, \dots, c_n\}.$$

Σ is called multi-valued dually Brouwerian (MVDB) iff any pair in $\Sigma \times \Sigma$ is nDB for some n (n is changing for different pairs, maybe).

Theorem 2.10 ([5]). *Let* $\mathbf{K}$ *be a variety. There is a natural number* $n \ge 1$ *such that the following conditions are equivalent:*

- *(i)* $\mathbf{K}$ *has DPC and CEP,*
- *(ii)* $\mathbf{K}$ *has nCS and CEP,*
- *(iii)* $\mathbf{K}$ *has nRCS,*
- *(iv)* $\mathbf{K}$ *has nREDPC,*
- *(v)* $\mathbf{K}$ *has nFDPC on subdirect products,*
- *(vi)* $\mathbf{K}$ *has nRAUPC,*
- *(vii) for every* $\mathcal{A} \in \mathbf{K}$ *the join-semilattice* $Comp(\mathcal{A})$ *of compact congruences is MVDB, moreover the generating set for* $Comp(\mathcal{A})$ *consisting of all principal congruences has the property: every pair of principal congruences is nDB in* $Comp(\mathcal{A})$.

3. A generalization of ideal congruencies

The idea of ideal congruences is due to R. Magari (see [6]): let $\mathcal{A} = \Pi(\mathcal{A}_j | j \in J_i)$ and let I be an ideal of the join-semilattice $\Pi(Comp(\mathcal{A}_i) | j \in J)$, where $Comp(\mathcal{A}_j)$ is the join-semilattice of compact congruences of $\mathcal{A}_j$. Then $a \equiv b(\theta_I^{\mathcal{A}})$ iff there is a $\theta = <\theta_j | j \in J> \in I$ satisfying $a_j \equiv b_j(\theta_j)$ for all $j \in J$, where $a = <a_j | j \in J>$ and $b = <b_j | j \in J>$. θ_I is called an ideal congruence.

Definition 3.1. *A variety* $\mathbf{K}$ *is said to have* *n ideal congruences for direct products* *(nICDP) iff for every* $\mathcal{A} \in \mathbf{K}$, *for any direct product representation* $\mathcal{A} = \Pi(\mathcal{A}_j | j \in J)$, *and for any congruence* θ *of* $\mathcal{A}$ *there exists a cover* $\sigma = \{J_1, \dots, J_n | J = J_1 \cup \dots \cup J_n\}$ *of* J *and there exist ideals* $I_1, \dots, I_n$, *respectively, in join-semilattices* $\Pi(Comp(\mathcal{A}_j) | j \in J_1), \dots, \Pi(Comp(\mathcal{A}_j) | j \in J_n)$ *such that*

$$\theta \vee \pi_{J_i} / \pi_{J_i} = \theta_{I_i} \quad \text{for all } i = 1, \dots, n,$$

where π_{J_i} *is the congruence of* $\mathcal{A}$ *induced by the projection homomorphism* $\mathcal{A} \xrightarrow{onto} \Pi(\mathcal{A}_j | j \in J_i)$; *we shall also say that* θ *is an n- ideal congruence.*

An analogue of 3.1 for subdirect products is as follows.

Definition 3.2. *A variety* $\mathbf{K}$ *is said to have n ideal congruences for subdirect products (nICSubDP) iff for every* $\mathcal{A}\in\mathbf{K}$, *for every subdirect representation* $\mathcal{A}\le\Pi(\mathcal{A}_j|j\in J)$, *and for any congruence* θ *of* $\mathcal{A}$ *there exists a cover* $\sigma= =\{J_1,\dots,J_n|J=J_1\cup\dots\cup J_n\}$ *of J and there exist ideals* $I_1,\dots,I_n$, *respectively, in join-semilattices* $\Pi(Comp(\mathcal{A}_j)|j\in J_1),\dots,\Pi(Comp(\mathcal{A}_j)|j\in J_n)$ *such that*

$$\theta\vee\pi_{J_i}\big|_{\mathcal{A}}/\pi_{J_i}\big|_{\mathcal{A}}=\theta_{\mathrm{I}_i}\big|_{\mathcal{A}_{J_i}}\quad \textit{for all } i=1,\dots,n,$$

where $\Phi|_X$ *is the restriction of* Φ *to* X *and* $\mathcal{A}_{J_i}$ *is the image of* $\mathcal{A}$ *under the projection homomorhism* $\Pi(\mathcal{A}_j|j\in J)\xrightarrow{onto}\Pi(\mathcal{A}_j|j\in J_i)$.

4. Characterizations of nICDP and nICSubDP

In the previous section we have defined our key notions and now we are ready to state first result.

Theorem 4.1. *The equivalent conditions (i)-(v) of Theorem 2.6 are equivalent to the following*

(vi) $\mathbf{K}$ *has nICDP.*

Proof. We shall prove the equivalence (2.6 (iv))↔(vi). Let $\mathbf{K}$ have nICDP and let $\mathcal{A}=\Pi(\mathcal{A}_j|j\in J)$, $a,b,c,d\in A$ with $c_j=d_j\left(\theta^{\mathcal{A}_j}(a_j,b_j)\right)$ for all $j\in J$. Then for $\theta(a,b)$ there is some cover $\sigma=\{J_1,\dots,J_n\}$ of J and there are some ideals $I_1,\dots,I_n$ of, respectively, $\Pi(Comp(\mathcal{A}_j)|j\in J_1)$, ..., $\Pi(Comp(\mathcal{A}_j)|j\in J_n)$, such that $\theta(a,b)\vee\pi_{J_i}/\pi_{J_i}=\theta_{\mathrm{I}_i}$ for all $i=1,\dots,n$. Consequently $a_{J_i}\equiv b_{J_i}(\theta_{\mathrm{I}_i})$ for all $i=1,\dots,n$. Next, there are $\theta_i=<\theta_{i,j}|j\in J_i>\in I_i$ such that $a_j\equiv b_j(\theta_{i,j})$ for $j\in J_i$, where $i=1,\dots,n$. So $\theta^{\mathcal{A}_j}(c_j,d_j)\le\theta^{\mathcal{A}_j}(a_j,b_j)\le\theta_{i,j}^{\mathcal{A}_j}$ for $j\in J_i$ and for every $i=1,\dots,n$. Thus

$$c_{J_1}\equiv d_{J_1}\left(\theta(a_{J_1},b_{J_1})\right),\ \dots,\ c_{J_n}\equiv d_{J_n}\left(\theta(a_{J_n},b_{J_n})\right).$$

Note: here we can even beforehand take $I_i=(<\theta(a_j,b_j)|j\in I_i>]$, for every $i=1,\dots,n$, a principal ideal.

Conversely, let $\mathbf{K}$ have nFDPC on direct products. Let $\mathcal{A}=\Pi(\mathcal{A}_j|j\in J)$ be a direct product representation of $\mathcal{A}$. Then for $a,b,c,d\in A$ with $c_j\equiv d_j\left(\theta(a_j,b_j)\right)$, $j\in J$, there exists some cover σ of J, so that for $\theta(a,b)$ we have:

$$\theta(a,b)\vee\pi_{J_i}/\pi_{J_i}=\theta_{\mathrm{I}_i},\quad i=1,\dots,n,$$

where $I_1=(<\theta(a_j,b_j)|j\in I_1>],\dots,I_n=(<\theta(a_j,b_j)|j\in I_n>]$, principal ideals. Thus every principal congruence of $\mathcal{A}$ is an n- ideal congruence.

Then, take some two elements $e,g\in A$. It is easy to show that

$$\theta(a,b)\vee\theta(e,g)\vee\pi_{J_i}/\pi_{J_i}=\theta_{\tilde{\mathrm{I}}_i},$$

where $\tilde{\mathrm{I}}_i=(<\theta(a_j,b_j)\vee\theta(e_j,g_j)|j\in J_i>]$ simultaneously for all $i=1,\dots,n$. For, clearly

$$\theta(a,b)\vee\theta(e,g)\vee\pi_{J_i}/\pi_{J_i}=\theta_{\mathrm{I}_i}\vee\left(\theta(e,g)\vee\pi_{J_i}/\pi_{J_i}\right)\le\theta_{\tilde{\mathrm{I}}_i},$$

so let u,v be an elements of A such that

$$u_{J_1}\equiv v_{J_1}\left(\theta_{\tilde{\mathrm{I}}_1}\right),\dots,\ u_{J_n}\equiv v_{J_n}\left(\theta_{\tilde{\mathrm{I}}_n}\right).$$

Consequently $u_j \equiv v_j\left(\theta(a_j,b_j)\vee\theta(e_j,g_j)\right)$ for all $j\in J$. Then there is a congruence scheme S_j over $\mathcal{A}'_j=\mathcal{A}_j/\theta(a_j,b_j)$ such that $\mathcal{A}'_j\models S_j(e'_j,g'_j,u'_j,v'_j)$, where x' denotes the image of x in $\mathcal{A}'_j$. So, if $J_{i_1}\cap J_{i_2}$ is nonempty then for all $j\in J_{i_1}\cup J_{i_2}$ these S_j's are automatically the same. Moreover, if some J_i is "isolated" from others, then we can give a common continuation of corresponding congruence schemes. In all cases, in fact, there works only one congruence scheme, so that $u\equiv v(\theta(a,b)\vee\theta(e,f))$.

The same arguments imply that every compact congruence of $\mathcal{A}$ is an n-ideal congruence. Since every congruence is a set union of compact ones, it follows readily that every congruence is an n-ideal congruence. This completes the proof of the theorem.

The analogy of 4.1 for subdirect products is as follows.

Theorem 4.2. *The equivalent conditions (i)-(vii) of Theorem 2.10 are equivalent to the following*

(viii) $\mathbf{K}$ *has nICSubDP.*

Proof. Let $\mathbf{K}$ have nCS+CEP. Then $\mathbf{K}$ has nICDP by 4.1 and thus CEP implies nICSubDP.

Conversely, suppose $\mathbf{K}$ has nICSubDP. Let $<\mathcal{A}_j,a_j,b_j,c_j,d_j>$, $j\in J$ be all algebras in $\mathbf{K}$, up to isomorphism, satisfying $c_j\equiv d_j\left(\theta(a_j,b_j)\right)$ for all $j\in J$, and $A_j=[a_j,b_j,c_j,d_j]$. Then we form $\Pi(\mathcal{A}_j|j\in J)$, $a=<a_j|j\in J>$, $b=<b_j|j\in J>$, $c=<c_j|j\in J>$, $d=<d_j|j\in J>$, and $A=[a,b,c,d]$, $\mathcal{A}$ a subalgebra (moreover, a subdirect representation in) $\Pi(\mathcal{A}_j|j\in J)$. Thus for $\theta^{\mathcal{A}}(a,b)$ there exists a cover σ of J and there exist ideals $I_1,\dots,I_n$ in, respectively, $\Pi(Comp(\mathcal{A}_j)|\, j\in J_1),\dots$, $\Pi(Comp(\mathcal{A}_j)|\, j\in J_n)$ such that

$$\theta^{\mathcal{A}}(a,b)\vee\pi_{J_i}|_{\mathcal{A}}/\pi_{J_i}|_{\mathcal{A}}=\theta_{1_i}\big|_{\mathcal{A}_{J_i}}\quad\text{for all } i=1,\dots,n,$$

where $\mathcal{A}_{J_i}$ is the image of $\mathcal{A}$ under the projection homomorphism onto $\Pi(\mathcal{A}_j|j\in J_i)$. Then $a_{J_1}\equiv b_{J_1}(\theta_{1_1}),\dots,$ and $a_{J_n}\equiv b_{J_n}(\theta_{1_n})$. Consequently there are $\theta_1\in I_1,\dots,\theta_n\in I_n$ such that $a_j\equiv b_j(\theta_{i,j})$ for all $j\in J_i, i=1,\dots,n$. Therefore $c_j\equiv d_j(\theta_{i,j})$ for all $j\in J_i$, $i=1,\dots,n$. Thus $c_{J_1}\equiv d_{J_1}(\theta_1),\dots,c_{J_n}\equiv d_{J_n}(\theta_n)$, so $c_{J_1}\equiv d_{J_1}(\theta_{1_1}),\dots,c_{J_n}\equiv d_{J_n}(\theta_{1_n})$. Now we sum up: we have verified that $<\mathcal{A}_{J_1},a_{J_1},b_{J_1},c_{J_1},d_{J_1}>,\dots,<\mathcal{A}_{J_n},a_{J_n},b_{J_n},c_{J_n},d_{J_n}>$ satisfy nRAUPC and so by 2.10, $\mathbf{K}$ satisfies all equivalent conditions (i)-(vii) of 2.10, completing the proof of the theorem.

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Received February 21, 2000; Revised April 5, 2000.
Translated by author.