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MAMEDOV Ya.D.

ON THE THEORY OF NONLOCAL BOUNDARY VALUE PROBLEM FOR LINEAR ABSTRACT ELLIPTIC EQUATION

Abstract

Nonlocal boundary value problem for linear abstract elliptic equation is considered and is giving formula for weak solution.

Let E be some Banach space. Consider nonlocal boundary value problem

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= Ax + f(t)\,(0\le t\le T) \\ x(0) &= x_0\,,\ x(t)=x(c) \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

with supposition, that linear constant is, generally speaking, non-bounded operator A such that its resolvent $R(\lambda)$ satisfies to condition:

$$\|R(\lambda)\| \le \frac{c}{1+|\lambda|} \quad (\lambda \le 0). \tag{3}$$

In this case equation (1) is called elliptic (see, for example, [1], [2], and also [3]). By $D(A)$ we define domain of definition, dense in E, of operator A.

1. At first, consider case when $f(t)\equiv 0$, i.e. homogeneous equation

$$\frac{d^2x}{dt^2} = Ax. \tag{4}$$

Definition 1. *Function $x(t)$ is called weak solution of equation (4), if:*

a) *it is continuous and has continuous second derivative on $(0,T)$, $A^{-1/2}x(t)$ has continuous first derivative on $[0,T]$;*
b) *the value of function $x(t)\,(0\le t\le T)$ belong to $D(A)$;*
c) *it satisfies to equation on interval $(0,T)$.*

Definition 2. *If weak solution $x(t)$ of equation (4) satisfies to boundary conditions (2), then we will call it as weak solution of boundary value problem (4), (2).*

It is known ([4]), that when operator A satisfies to condition (3), then for this operator fractional powers could be defined, in particular, as $A^{-1/2}$.

Introduce denotations

$$y(t) = A^{-1/2}\frac{dx}{dt},$$

equation (4) could be rewritten in the form of system

$$\left.\begin{aligned} \frac{dx}{dt} &= A^{1/2}y, \\ \frac{dy}{dt} &= A^{1/2}x. \end{aligned}\right\}$$

And if we introduce substitution

$$z=\frac{x-y}{2},\quad \vartheta=\frac{x+y}{2},$$

then for this functions we obtain equation

$$\left.\begin{aligned}\frac{dz}{dt}&=A^{1/2}z,\\ \frac{d\vartheta}{dt}&=A^{1/2}\vartheta.\end{aligned}\right\}$$

Operator $-A^{-1/2}$ is generated operator of analytic half group $V(t)$, satisfying to C_0 - condition ([5]). Therefore for first equation the Cauchy problem is uniformly correct, and for second one is uniformly correct inverse Cauchy problem consequently,

$$z(t)=V(t)z_0\ ,\quad \vartheta(t)=V(T-t)\vartheta_T\,.$$

If $z_0,\vartheta_T\in D\left(A^{1/2}\right)$, then function

$$x(t)=V(t)z_0+V(T-t)\vartheta_T$$

is weak solution of equation (4) ([1]).

Now consider boundary value problem (4), (2) and define its weak solution.

Introduce denotations

$$g_1=A^{-1/2}z_0\ ,\quad g_2=A^{-1/2}\vartheta_T\,.$$

Any solution of equation (4) could be determined by formula

$$x(t)=V_1(t)g_1+V_2(t)g_2(g_1,g_2\in E),\tag{5}$$

where

$$V_1(t)=V(t)A^{-1/2}\ ,\quad V_2(t)=V(T-t)A^{-1/2}\,.$$

We have

$$\left.\begin{aligned}V_1(0)g_1+V_2(0)g_2&=x_0\ ,\\ V_1(T)g_1+V_2(T)g_2&=V_1(c)g_1+V_2(c)g_2\end{aligned}\right\}$$

$$\left.\begin{aligned}V_1(0)g_1+V_2(0)g_2&=x_0\ ,\\ \left[V_1(T)-V_1(0)\right]g_1&=\left[V_2(T)+V_2(0)\right]g_2=0\end{aligned}\right\}$$

By

$$D=\begin{vmatrix}V_1(0) & V_2(0)\\ V_1(T)-V_1(c) & V_2(T)-V_2(c)\end{vmatrix}$$

we denote operator determinant of this system. From this system we could define g_1,g_2:

$$\left.\begin{aligned}Dg_1&=\left[V_2(T)+V_2(c)\right]x_0\ ,\\ Dg_2&=\left[V_1(T)-V_1(c)\right]x_0\,.\end{aligned}\right\}\tag{6}$$

If we suppose, that operators

$$D^{-1}\left[V_i(T)-V_i(c)\right]\ (i=1,2)$$

are determined on whole space E, then from (6) we can determine g_1 and g_2. Determined elements we substitute into (5), and we obtain weak solution of problem (4), (2).

2. Now consider non-homogeneous equation (1). We suppose, that $f(t)\ (0\le t\le T)$ is continuous function. By substitutions

$$y=A^{-1/2}\frac{dx}{dt},\ z=\frac{x-y}{2},\ \vartheta=\frac{x+y}{2}$$

equation (1) reduces to the system

$$\left.\begin{aligned}\frac{dz}{dt}=-A^{1/2}+\frac{1}{2}A^{-1/2}f(t),\\ \frac{d\vartheta}{dt}=A^{1/2}\vartheta-\frac{1}{2}A^{-1/2}f(t)\end{aligned}\right\}\ (0\le t\le T).$$

Solutions of these equations, correspondingly, have the form ([1])

$$z(t)=V(t)z_0+\frac{1}{2}\int_0^t V(t-\tau)A^{-1/2}f(\tau)d\tau,$$

$$\vartheta(t)=V(T-t)\vartheta_T+\frac{1}{2}\int_t^T V(\tau-t)A^{-1/2}f(\tau)d\tau .$$

Then general solution of equation (1) have the form

$$x(t)=V(t)z_0+V(T-\tau)\vartheta_T+\frac{1}{2}\int_0^t V(|t-\tau|)A^{-1/2}f(\tau)d\tau . \tag{7}$$

First two members in this form are general solution of homogeneous equation, which was considered in previous item.

If we suppose, that function $f(t)$ satisfies to Hölder's condition, then as it was proved in [1], function

$$g(t)=\int_0^T V_0(t,\tau)f(\tau)d\tau,$$

will be weak solution of equation (1). Here

$$V_0(t,\tau)=\frac{1}{2}V(|t-\tau|)A^{-1/2}=\begin{cases}\frac{1}{2}V(t-\tau)A^{-1/2} \text{ for } t\ge\tau,\\ \frac{1}{2}V(\tau-t)A^{-1/2} \text{ for } t\le\tau\end{cases}$$

is fundamental solution of equation (1) and have the known property ([1]).

Taking account of denotations from first item, we can rewrite formula (7) in the following form

$$x(t)=V_1(t)g_1+V_2(t)g_2+g(t). \tag{8}$$

Taking account of boundary conditions (2), from (8) we obtain

$$\left.\begin{aligned}V_1(0)g_1+V_2(0)g_2+g(0)=x_0,\\ V_1(T)g_1+V_2(T)g_2+g(T)=V_1(c)g_1+V_2(c)g_2+g(c)\end{aligned}\right\}$$

or,

$$\left.\begin{aligned}V_1(0)g_1+V_2(0)g_2=x_0-g(0),\\ [V_1(T)+V_1(c)]g_1+[V_2(T)+V_2(c)]g_2=g_1(c)-g_1(T)\end{aligned}\right\}$$

Solving this system, we have

$$\left.\begin{aligned}Dg_1=[V_2(T)-V_2(c)][x_0g(0)]-V_2(0)[g(c)-g(T)],\\ Dg_2=-V_1(0)[g(T)-g(c)]-[V_1(T)-V_1(c)][x_0-g(0)].\end{aligned}\right\}$$

If these formulas we will take into account in (8), we obtain

$$Dx=V_1(t)\{[V_2(T)-V_2(c)][x_0-g(0)]+V_2(0)[g(T)-g(c)]\}-$$
$$-V_2(t)\{[V_1(T)-V_1(c)][x_0-g(0)]+V_1(0)[g(T)-g(c)]\}+g(t),$$
$$Dx=V_1(t)[V_2(T)-V_2(c)]x_0-V_2(t)[V_1(T)-V_1(c)]x_0-V_1(t)[V_2(T)-V_2(c)]g(0)+$$
$$+V_2(t)[V_1(T)-V_1(c)]g(0)+V_1(t)V_2(0)[g(T)-g(c)]-V_2(t)V_1(0)[g(T)-g(c)]+Dg(t),$$

$$Dx = V_1(t)[V_2(T) - V_2(c)]x_0 - V_2(t)[V_1(T) - V_1(c)]x_0 + \int_0^T G_0(t,s)f(s)ds, \qquad (9)$$

where

$$G_0(t,s) = \{V_2(t)[V_1(T) - V_1(c)] - V_1(t)[V_2(T) - V_2(c)]\}V_0(T,s) +$$
$$+ [V_1(t)V_2(0) - V_2(t)V_1(0)][V_0(T,s) - V_0(c,s)]$$

or,

$$G_0(t,s) = \begin{vmatrix} V_1(t) & V_2(t) & V_0(t,s) \\ V_1(0) & V_2(0) & V_0(T,s) \\ V_1(T) - V_1(0) & V_2(T) - V_2(c) & V_0(T,s) - V_0(c,s) \end{vmatrix}.$$

Now suppose, that

$$D^{-1}V_i(0), \quad D^{-1}[V_i(T) - V_i(c)] \ (i = 1,2)$$

are determined in whole space E, then from (9) we have

$$x(t) = D^{-1}\{V_1(t)[V_2(T) - V_2(c)] - V_2(t)[V_1(T) - V_1(c)]\}x_0 + \int_0^T G(t,s)f(s)ds,$$

where

$$G(t,s) = D^{-1}G_0(t,s).$$

And this exactly is weak solution of problem (1), (2).

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Mamedov Ya.D.

Baku State University named after M. Rasulzadeh.
23, Z.I. Khalilov str., 370148, Baku, Azerbaijan.

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