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EXACT SOLUTION OF YANG-MILLS SELF-DUALITY EQUATIONS

Abstract

The exact $A_1(SL(2,R))$, $A_2(SL(3,R))$- solutions of self- duality equations are found. In the special case the $O(4)$- invariant with instanton number equal to one arises.

In the last few years a great interest has been paid [1-8] on investigation of self-dual Yang- Mills equation because it has been shown that a large number of one, two and $(1+2)$- dimensional integrable models can be obtained from it by symmetry reduction and by imposing the constraints on Yang- Mills potentials. The universality of the self-dual Yang- Mills model as an integrable system has been confirmed in the recent paper [9] where the general scheme of the reduction of the Belavin- Zakharov Lax pair for self-duality [10] has been represented over an arbitrary subgroup from the conformal group of transformations of R_4- space. As the result of this reduction one has the Lax pair representation for the corresponding differential equations of a lower dimension.

In the Leznov- Saveliev approach [11] the cylindricall symmetrric configurations of Yang- Mills fields are considered, that is the solutions are invariant with respect to the so- called diagonal group, the generators of which are composed of the generators of a subgroup ofthe conformal group of coordinate transformations and the $SU(2)$ - subgroup of the gauge group. In this case the number of different two-dimensional reductions is defined by nonequivalent embeddings of the $SU(2)$- group into the gauge group. In particular, an exactly integrable system of generalized Toda lattice was derived and its general solution was obtained.

In this work, following the Leznov- Mukhtarov approach [12], we construct the solution of self-duality equation depending on r - independent linear self-dual systems, each of which contains $2\omega_\alpha+1$ members, where ω_α are the indexes of the semisimple algebra. The $O(4)$- invariant solution, having no singularities in the whole four dimensional space, aries when the solutions of chains of linear systems are simply numerical constants.

1. From the group of motion of four dimensional space $(x_1,x_2,x_3,x_4)\in R(4)$ let us choose the group $SU(2)$ which transforms the pair complex coordinates y,z $(y=x_1+ix_2,\ z=x_3+ix_4)$ as the components of the two- dimensional (spinor) representation of this group. The components of conjugated spinor $(\bar z,-\bar y)$ and any linear combination of the form $(y+\lambda\bar z,\ z-\lambda\bar y)$ are transformed at the same way. The infinitesimal operators of this algebra $L_\pm, L_0(H)$ have the form

$$L_+=y\frac{\partial}{\partial z}-\bar z\frac{\partial}{\partial \bar y},\quad L_-=z\frac{\partial}{\partial y}-\bar y\frac{\partial}{\partial \bar z},\quad L_0=y\frac{\partial}{\partial y}-z\frac{\partial}{\partial z}+\bar z\frac{\partial}{\partial \bar z}-\bar y\frac{\partial}{\partial \bar y}. \qquad (1)$$

The algebra $sl(2,R)$ may be embedded in gauge algebra in many unequivalent ways. In this work we shall use the so-called principal embedding, the infinitesimals operators of which are defined as:

$$J_\pm=\sum_{\alpha=1}^{r}\sqrt{\omega_\alpha}X_\alpha^\pm,\quad H=\sum_{\alpha=1}^{r}\omega_\alpha h_\alpha, \qquad (2)$$

where X_α, h_α are the generators of the simple roots and Cartan elements of a semisimple algebra, $\omega_\alpha = 2\sum_{\beta=1}^{r}(k^{-1})_{\alpha\beta}$ is the set of its indexes, k is the Cartan matrix and r is its rank. For more details on the principle embedding, see for instance [4] (there it is called minimal).

The generators of a semisimple algebra can be decomposed into 2 multiplets, i.e., they may be marked by the quantum numbers (l,m) of irreducible self- dual equations, which are invariant with respect to the total momentum $\vec{S} = \vec{J} + \vec{L}$ (see (1-2)). For this purpose we need our element F_0 from the algebra which satisfies the condition $[F_0, \vec{S}] = 0$.

To construct such an element let us notice that from the components of a spinor we may construct $(2l+1)$- ordered vectors $\xi_\alpha^l = (y + \lambda\bar{z})^{2l-\alpha}(z - \lambda\bar{y})^\alpha$ $(0 \le \alpha \le 2l)$ which are transformed according to the $(2l+1)$- dimensional representation of the algebra $sl(2,R)$. Constructing these vectors with generators of gauge algebra, having the same quantum numbers, we get the invariant F_0^l connected with the l - multiplet.

For instance, three basis vectors $(z-\lambda\bar{y})^2, (z-\lambda\bar{y})(y+\lambda\bar{z}), (y-\lambda\bar{z})^2$ are transformed according to vector representation of the $sl(2,R)$ algebra and the corresponding invariant has the form:

$$F_0^1 = (y+\lambda\bar{z})^2 J_- - (y+\lambda\bar{z})(z-\lambda\bar{y})H - J_+(z-\lambda\bar{y})^2$$

as can verified by direct calculations.

In general, $F_0 = \sum_{\alpha=1}^{r} c_\alpha F_0^\alpha$, where summation is over all multiplets of the algebra.

As F_0 is invariant with respect to the total momentum it is always possible to transform F_0 to the coordinate system where $y = \bar{y} = 0$, $z = \bar{z} = \sqrt{R}$, $R = z\bar{z} + y\bar{y}$; so F_0 may be presented in the form

$$F_0 = T\left(\exp(J_-\lambda)\sum_{\alpha=1}^{r} c_\alpha R^{\omega_\alpha} J_+^\alpha \exp(-J_-\lambda)\right)T^{-1}, \tag{3}$$

where

$$T = \exp\left(-J_+\frac{\bar{y}}{\bar{z}}\right)\exp\left(J_-\frac{y\bar{z}}{R}\right)\exp\left(H\ln\frac{\sqrt{R}}{\bar{z}}\right),$$

J_+^α is highest vector of α - multiplet, having the properties

$$[H, J_+^\alpha] = 2\omega_\alpha J_+^\alpha, \quad [J_+^\alpha, J_+^\beta] = 0.$$

The last commutativity relations of the highest vectors belonging to different multiplet distinguish the principal embedding from the others.

2. Let us write the self- dual equations in Yang's form

$$G^{-1}G_{\bar{y}} = -f_z, \; G^{-1}G_{\bar{z}} = -f_y, \tag{4}$$

where the elements G and f take values in the gauge group and gauge algebra respectively.

To integrate equation (4) it is necessary to solve the homogeneous Riemann problem

$$e^{F_0}\Omega_c^+ = \Omega_c^-, \tag{5}$$

where $\Omega_c^{\pm}$ are the boundary values of analytical functions, taking values in the gauge group, and are defined outside and inside of the contour c, respectively; F_0 is an arbitrary function of three independent variables $F_0 \equiv F_0(\lambda, y+\lambda\bar{z}, z-\lambda\bar{y})$ taking values in the gauge algebra. The boundary condition for (5) is $\Omega^+ \to 1+\frac{F}{\lambda}$ when $\lambda \to \infty$. The point $\lambda = 0$ is inside and the point $\lambda = \infty$ is outside the contour c. Using the usual methods for the Riemann problem it may be proved that equation (4) are fulfilled if we take $G = \Omega^-|_{\lambda=0}$, $f = F$, where Ω^-, F are taken from the solution of Riemann problems.

If we want to find the solution of the self-dual equations, which are invariant with respect to the total momentum S, we must take F_0 in form (3), where c_α are arbitrary functions of $\lambda : c_\alpha = c_\alpha(\lambda)$.

Here we shall consider the more general case supposing that $c_\alpha \equiv$ $\equiv c_\alpha(\lambda, y+\lambda\bar{z}, z-\lambda\bar{y})$. The same arguments will be used for the function F_0.

Now we shall solve the Riemann problem for this case. First of all, we include the element T, which is λ independent and rewrite (5) in the form

$$\left(\exp\sum c_\alpha J_\alpha^+\right)e^{-\lambda J^-}\Omega_c^+ = e^{-\lambda J^-}\Omega_c^-.$$

From this moment we include the factor R^{ω_α} (see (3)) in $c_\alpha : c_\alpha R^{\omega_\alpha} \to c_\alpha$.

Let us perform some identical transformations:

$$\left(\exp\sum c_\alpha J_\alpha^+\right)e^{-\lambda J^-} = \left(\exp\sum \tilde{c}_\alpha J_\alpha^+\right)e^{J^+/\lambda}e^{-\lambda J^-} = \left(\exp\sum \tilde{c}_\alpha J_\alpha^+\right)Me^{H\ln\lambda}e^{-J^+/\lambda},$$

$$\tilde{c}_\alpha = c_\alpha - \delta_{1\alpha}/\lambda, \qquad M^{-1}J_\alpha^+ M = (-1)^\alpha J_\alpha^-.$$

Taking into account the commutativity of the highest vectors $\left[J_+^\alpha, J_+^\beta\right] = 0$ and using the Sokhotsky- Plemeli formulae, we have

$$\left(\left(\exp\sum c_\alpha^+ J_\alpha^-\right)e^{H\ln\lambda}e^{-J^+/\lambda}\Omega^+\right)_c = \left(\left(\exp\sum c_s^- J_s^-\right)Me^{-\lambda J^-}\Omega^-\right)_c \equiv p(\lambda), \qquad (6)$$

where $c_\alpha^\pm(\lambda) = \int \frac{d\lambda'}{\lambda'-\lambda}c_\alpha(\lambda') : \left(\left(c_\alpha^+(\lambda)\right)_c - \left(c_\alpha^-(\lambda)\right)_c = \tilde{c}_\alpha(\lambda)\right)$. All factors of the left-hand side of (6) are analytical outside the integration contour and the ones on the right-hand side-inside it. Thus, from the Liouville theorem we may conclude that the function $p(\lambda)$ taking values in the group has in any representation the matrix elements, which are polynomial over λ in the hole complex plane.

The asymptotic condition $\Omega^+ \to 1+\frac{F}{\lambda}$ allows us to find the coefficients of polynomials of each element and to solve the self-dual equations in the case under consideration.

3. To make the situation clearer we shall use the method of the previous section in the simplest case of the $A_1(SL(2,R))$ gauge algebra. Let's parameterize the group element $e^{-J_+/\lambda}\Omega^+$ by Euler's angles: $e^{-J_+/\lambda}\Omega^+ = \exp\alpha J_+ \exp\tau H \exp\beta J_-$ and take the elements $J_\pm$, H in two-dimensional representation. Under these assumptions equation (6) takes the form:

$$\begin{pmatrix} \lambda e^\tau & \lambda\beta e^\tau \\ \lambda\left(c_1^+ + \frac{1}{\lambda^2}\alpha\right)e^\tau & \lambda\left(c_1^+ + \frac{1}{\lambda^2}\alpha\right)\beta e^\tau + \frac{1}{\lambda}e^{-\tau} \end{pmatrix} = p(\lambda) = \begin{pmatrix} \lambda+\tau_0 & \beta_0 \\ c_0 & 0 \end{pmatrix}, \qquad (7)$$

where we have used the asymptotic expansion of the Cauchy integral at the infinite point

$$\theta=\frac{1}{2\pi i}\int_c \frac{\theta(\lambda')d\lambda'}{\lambda'-\lambda}=\frac{\theta_0}{\lambda}+\frac{\theta_1}{\lambda^2}+\cdots+\frac{\theta_s}{\lambda^{s+1}}+\cdots,\ \theta_s=\frac{1}{2\pi i}\int_c (\lambda')^s\theta(\lambda')d\lambda'$$

and the asymptotical conditions $e^{-J_+/\lambda}\Omega^+ \to 1+\frac{F-J_+}{\lambda}+\ldots$, i.e.

$$e^{\tau} \underset{\lambda\to\infty}{=} 1+\frac{\tau_0}{\lambda}+\cdots,\qquad \alpha \underset{\lambda\to\infty}{=} \sum_{s=0}^{\infty}\frac{\alpha_s}{\lambda^{s+1}},\qquad \beta \underset{\lambda\to\infty}{=} \sum_{s=0}^{\infty}\frac{\beta_s}{\lambda^{s+1}}.$$

From (7) w have

$$e^{\tau}=1+\frac{\tau_0}{\lambda},\quad \alpha=\lambda^2\left(\frac{c_0}{\lambda+\tau_0}-c_1^+\right)=-(c_1+c_0\tau_0)+\frac{c_0\tau_0^2-c_2}{\lambda}+\cdots$$

i.e.

$$\tau_0=-\frac{c_1}{c_0},\qquad \alpha_0=-c_2+\frac{c_1^2}{c_0}.$$

In the same way we get $\beta_0=-\frac{1}{c_0}$ and for f we have:

$$f=J_+\beta_0+H\tau_0+J_-\alpha_0=J_+\left(1-\frac{1}{c_0}\right)-H\frac{c_1}{c_0}+J_-\left(-c_2+\frac{c_1^2}{c_0}\right). \tag{8}$$

After the necessary transformation $f\to T^{-1}fT$ (see (3)) and some trivial gauge transformation f turns into the Hooft solution in its usual form:

$$f=J_+\frac{1}{\varphi_0}-H\frac{\varphi_1}{\varphi_0}+J_-\left(-\varphi_2+\frac{\varphi_1^2}{\varphi_0}\right),$$

where $\varphi_0,\varphi_1,\varphi_2$ are the terms of the chain of the self-dual linear equations which are connected with c_0,c_1,c_2 (from (3) and see after (7)) by the relations

$$\varphi_0=c_0+\frac{1}{R},\ \ \varphi_1=c_1+\frac{y}{\bar z}\frac{1}{R},\ \ \varphi_2=c_2+\left(\frac{y}{\bar z}\right)^2\frac{1}{R}. \tag{9}$$

The instanton charge density q for this solution is

$$q\sim\Box\Box\ln\left(1-c_0R^2\right).$$

4. In this section we shall give the explicit form of the solution of equation (4) for the case of the algebra A_2. We keep notation c_s for the terms of length three chains (c_0,c_1,c_2) and d_s for the terms of length five chain (d_0,d_1,d_2,d_3,d_4). In these notations

$$f=\begin{pmatrix} p_1 & y_1-2 & \mu_1 \\ -c_2-c_1p_1-c_0p_0 & -c_1y_1-c_0y_0 & -c_1\mu_1-c_0\mu_0-2 \\ -d_4-\left(d_3-\frac{(c_1)^2}{2}\right)p_1- & -\left(d_3-\frac{(c_1)^2}{2}\right)y_1- & -\left(d_3-\frac{(c_1)^2}{2}\right)\mu_1- \\ -(d_2-c_0c_1)p_0 & -(d_2-c_0c_1)y_0-c_2 & -(d_2-c_0c_1)\mu_0 \end{pmatrix},$$

where the pairs (p_0,p_1), (y_0,y_1), (μ_0,μ_1) are the solutions of linear systems of two equations:

$$\begin{pmatrix} d_1+\frac{(c_0)^2}{2} & d_0 \\ d_2 & d_1-\frac{(c_0)^2}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} d_3+c_0c_1 \\ d_3+\frac{(c_1)^2}{2} \end{pmatrix}, \quad -\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}, \quad -\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The right hand side of these equations corresponds to the pairs (p_0, p_1), (y_0, y_1), (μ_0, μ_1) respectively.

We see that f may have singularities in the finite points of fundamental space only when the set of the system is zero: $Det = (d_1)^2 R^4 - d_0 d_2 R^4 - \frac{(c_0 R - 1)^4}{4}$.

Writing the expression for the determinant we have in mind the dependence of c_α on R, which we have included in the determination of c_α.

If all terms of the chains are constants, we have, as it was mentioned above, $O(4)$ invariant solution of (4). In this case if we choose $c_0 < 0$ and $(d_1)^2 - d_0 d_2 < 0$ we have the constant sign of the determinant and the solution of self-dual equation (4) in the case of algebra A_2 has no singularities at all points of four- dimensional space.

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