MUSHTAGOV F.M.

THE UNIQUENESS OF SOLUTION OF CAUCHY PROBLEM FOR THE SECOND ORDER PARABOLIC EQUATIONS WITH NONUNIFORM DEGREE DEGENERATION

Abstract

An article deals with Cauchy problem for a class of the second order parabolic equations with nonuniform degree degeneration. The uniqueness of solution in a class of slow growing Tikhnov type functions has been proved.

Let R_{n+1} be $(n+1)$- dimensional Euclidean space of points $(x,t)=(x_1,...,x_n,t)$, $R_{n+1}^+ = R_{n+1} \cap \{(x,t): t>0\}$, $S_T = R_{n+1}^+ \cap \{(x,t): t<T\}$, where $T \in (0,\infty)$. We consider Cauchy problem in R_{n+1}^+

$$Lu = \sum_{i,j=1}^n a_{ij}(x,t)u_{ij} - u_t = 0, \quad (x,t)\in S_T; \quad u|_{t=0} = f(x). \tag{1}$$

We suppose, that for all $(x,t)\in S_T$ and for any n- dimensional vector $\xi=(\xi_1,...,\xi_n)$ the next condition is fulfilled

$$\mu\sum_{i=1}^n \lambda_i(x,t)\xi_i^2 \le \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \le \mu^{-1}\sum_{i=1}^n \lambda_i(x,t)\xi_i^2 , \tag{2}$$

where $\mu\in(0,1]$ is constant, $\lambda_i(x,t)=\left(1+|x|_\alpha+\sqrt{t}\right)^{-\alpha_i}$, $|x|_\alpha = \sum_{i=1}^n |x_i|^{\frac{2}{2-\alpha_i}}$, $\alpha=(\alpha_1,...,\alpha_n)$, $\alpha_i\in[0,2)$; $i=1,...,n$. We denote through u_i $\frac{\partial u}{\partial x_i}$ and through u_{ij} $\frac{\partial^2 u}{\partial x_i \partial x_j}$ for $i,j=1,...,n$. Moreover without loss of generality we can assume that $a_{ij}(x,t)=a_{ji}(x,t)$. The aim of this paper is to find a class of uniqueness Cauchy problem (2). We note, that for the heat equation the uniqueness solution of Cauchy problem in the class of slowgrowing functions was established by A.N Tikhonov. [1] and S. Taclind [2]. This result was carried to the arbitrary uniforms second order parabolic equations of the nondivergent structure in [3] (look also [4]-[5]). More complete review of literature by subjects of researches it is possible to find in monographies [6] and [7].

At first we agree in some denotes and definitions. For n-dimensional vector x^0 and positive numbers R and k we denote through $E_R^{x^0}(k)$ next ellipsoid $\left\{x: \sum_{i=1}^n \frac{(x_i-x_i^0)^2}{R^{-\alpha_i}} < (kR)^2\right\}$ and through $C_{x^0,R}^{t^1,t^2}(k)$ next $(n+1)$- dimensional cylinder $E_R^{x^0}(k)\times(t^1,t^2)$. Everywhere further $\Gamma(D)$ is parabolic boundary of the domain D and writing $C(...)$ denotes, that positive constant C depends only on contain of parentheses.

For positive numbers R, s, β we set

$$F_{s,\beta}^{(R)}(x,t)=\begin{cases} t^{-s}\exp\left[-\sum_{i=1}^{n}\frac{x_i^2}{R^{-\alpha_i}}\Big/4\beta t\right], & \text{if } t>0 \\ 0, & \text{if } t\le 0 \end{cases}$$

Definition 1. *Function $\vartheta(x,t)$, defined in layer S_T, is called α-slow growing, if there are constants $c_1>0$, $c_2>0$ such that for all positive R*

$$\sup_{C_{0,R}^{0,T}(1)}|\vartheta(x,t)|\le C_1 e^{c_2R^2}.$$

Definition 2. *Function $\vartheta(x,t)$, defined in layer S_T, is called L- subparabolic (L- superparabolic), if $\vartheta(x,t)\in C^{2,1}(S_T)\cap C\{(x,t):t=0\}\cap C\{(x,t):t=T\}$ and $L\vartheta(x,t)\ge 0(\le 0)$ for $(x,t)\in S_T$.*

Lemma 1. *Let $A_1^R=\left(E_R^0(2)\setminus E_R^0(1)\right)\times\left(0,R^2\right)$. If coefficients of operator L, satisfy to condition (2), then there are $S(\mu,\alpha,n)$ and $\beta(\mu,\alpha,n)$ such that for $R\ge 1$, $(y,\tau)\in A_1^R$*

$$L(x,t)F_{s,\beta}^{(R)}(x-y,t-\tau)\le 0;\ \ (x,t)\in A_1^R\setminus\{(y,\tau)\}. \qquad (3)$$

Proof. Taking into account (2) we have

$$LF_{s,\beta}^{(R)}=F_{s,\beta}^{(R)}\left\{\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{(x_i-y_i)(x_j-y_j)}{4\beta^2(t-\tau)^2R^{-\alpha_i-\alpha_j}}-\frac{1}{2\beta(t-\tau)}\sum_{i=1}^{n}\frac{a_{ii}(x,t)}{R^{-\alpha_i}}+\frac{S}{t-\tau}-\right.$$

$$\left.-\frac{\sum_{i=1}^{n}(x_i-y_i)^2/R^{-\alpha_i}}{4\beta(t-\tau)^2}\right\}\le F_{s,\beta}^{(R)}\left\{\frac{1}{4\mu\beta^2(t-\tau)^2}\sum_{i=1}^{n}\lambda_i(x,t)\frac{(x_i-y_i)^2}{R^{-2\alpha_i}}-\frac{n\mu}{2\beta(t-\tau)}\times\right. \qquad (4)$$

$$\left.\times\sum_{i=1}^{n}\frac{\lambda_i(x,t)}{R^{-\alpha_i}}+\frac{s}{t-\tau}-\frac{\sum_{i=1}^{n}(x_i-y_i)^2/R^{-\alpha_i}}{4\beta(t-\tau)^2}\right\}$$

Since $x\in E_R^0(2)$, then $\sum_{i=1}^{n}\frac{x_i^2}{R^{-\alpha_i}}<\left(2R^2\right)$ and therefore for $i=1,\dots,n$ $|x_i|<2R^{1-\alpha_i/2}$, $|x_i|^{\frac{2}{2-\alpha_i}}\le 2^{\frac{2}{2-\alpha^+}}R=C_3(\alpha)R$, where $\alpha^+=\max\{\alpha_1,\dots,\alpha_n\}$.

Consequently

$$|x|_\alpha\le nC_3R. \qquad (5)$$

Moreover, because of $t\in\left(0,R^2\right)$, then

$$\sqrt{t}\le R. \qquad (6)$$

From (5) and (6) we get, that for $R\ge 1$

$$1+|x|_\alpha+\sqrt{t}\le(2+nC_3)R.$$

Frow the last we obtain

$$\lambda_i(x,t)\ge(2+nC_3)^{-\alpha^+}R^{-\alpha_i}=C_4(\alpha,n)R^{-\alpha_i};\ \ i=1,\dots,n. \qquad (7)$$

Other wise because of $x \notin E_R^0(1)$, then $\sum_{i=1}^{n} \frac{x_i^2}{R^{-\alpha_i}} \geq R^2$. From here we obtain existing i_0, $1 \leq i_0 < n$ such that $\left|x_{i_0}\right| \geq \frac{R^{1-\alpha_{i0}/2}}{n}$, i.e. $\left|x_{i_0}\right|^{\frac{2}{2-\alpha_{i_0}}} \geq \frac{R}{n^{\frac{2}{2-\alpha^+}}} = C_5(\alpha,n)R$.

Hence

$$\left|x\right|_\alpha \geq \left|x_{i_0}\right|^{\frac{2}{2-\alpha_{i_0}}} \geq C_5 R. \tag{8}$$

From (8) we deduce, that for $R \geq 1$

$$1 + \left|x\right|_\alpha + \sqrt{t} \geq C_5 R_n,$$

and thus

$$\lambda_i(x,t) \leq C_5^{-\alpha^-} R^{-\alpha_i} \leq C_6(\alpha,n) R^{-\alpha_i}; \quad i = 1,\ldots,n, \tag{9}$$

where $\alpha^- = \min\{\alpha_1,\ldots,\alpha_n\}$. From (7) and (9) we conclude that for $(x,t) \in A_1^R$

$$C_4 \leq \frac{\lambda_i(x,t)}{R^{-\alpha_i}} \leq C_6; \quad i = 1,\ldots,n. \tag{10}$$

Taking into account (10) in (4) we get

$$LF_{s,\beta}^{(R)} \leq F_{s,\beta}^{(R)} \left\{ \frac{1}{4\beta(t-\tau)^2} \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{R^{-\alpha_i}} \left[\frac{C_6}{\mu\beta} - 1 \right] + \frac{1}{t-\tau} \left[5 - \frac{n^2 C_4 \mu}{2\beta} \right] \right\}.$$

Now choosing $\beta = \frac{C_6}{\mu}$; $s = \frac{n^2 C_4 \mu^2}{2C_6}$ we obtain demanding estimate (3). Lemma is proved.

Let $E = \min\left\{T, \frac{1}{64 C_2 \beta \lambda^2}\right\}$, where $\lambda = 2^{\frac{2}{2-\alpha^+}}$, β is constant of lemma 1.

Lemma 2. *Let $u(x,t)$ be L- subparabolic in S_E function, nonpositive for $t = 0$. So that, if $u(x,t) - \alpha$ - slow growing function, then*

$$\overline{\lim_{x \to \infty}} u(x,t) \leq 0.$$

Proof. For arbitrary $R > 1$ we consider subsidiary function

$$\vartheta_R(x,t) = M_1 e^{C_2 R^2} \int_{\partial E_R^0(1)} (t+\varepsilon)^{-s} \exp\left[-\frac{\sum_{i=1}^{n} (x_i - \xi_i)/R^{-\alpha_i}}{4\beta(t+\varepsilon)} \right] ds_\xi +$$

$$+ M_1 e^{C_2 \lambda^2 R^2} \int_{\partial E_R^0(2)} (t+\varepsilon)^{-s} \exp\left[-\frac{\sum_{i=1}^{n} (x_i - \xi_i)/R^{-\alpha_i}}{4\beta(t+\varepsilon)} \right] ds_\xi,$$

where s and β are chosen according to previous lemma, but positive constant M_1 will be defined later.

According to lemma 1 function $\vartheta_R(x,t)$ is L- superbarabolic in $B_\varepsilon^R = A_1^R \cap \cap \{(x,t): 0 < t < \varepsilon\}$. At the lower base of domain B_ε^R (for $t = 0$) $\vartheta_R(x,t) > 0$

For $(x,t) \in \partial E_R^0(1) \times (0,\varepsilon)$

$$\vartheta_R(x,t) \geq \frac{M_1 e^{C_2R^2}}{(2\varepsilon)^s} \int\limits_{\partial E_R^0(1)} \exp\left[-\frac{\sum\limits_{i=1}^{n}(x_i-\xi_i)^2/R^{-\alpha_i}}{4\beta\varepsilon}\right] ds_\xi . \qquad (11)$$

We fix $x \in \partial E_R^0(1)$ and denote through E_1^+ set $\left\{\xi : \xi \in \partial E_R^0(1), \sum\limits_{i=1}^{n}\frac{(x_i-\xi_i)}{R^{-\alpha_i}} \leq 4\beta\varepsilon\right\}$. It is clear, that exists $R_1(\mu,\alpha,n)$ such that for $R \geq R_1$ $mes_{n-1}(E_1^+) \geq 1$.

Then

$$\int\limits_{\partial E_R^0(1)} \exp\left[-\frac{\sum\limits_{i=1}^{n}\frac{(x_i-\xi_i)^2}{R^{-\alpha_i}}}{4\beta\varepsilon}\right] ds_\xi \geq \int\limits_{E_1^+} e^{-1} ds_\xi \geq e^{-1}$$

that together with (11) give

$$\vartheta_R(x,t) \geq \frac{M_1 e^{C_2R^2}}{e(2\varepsilon)^s}.$$

If choose and fix $M_1 = C_1 e(2\varepsilon)^s$, then $\vartheta_R(x,t) \geq C_1 e^{C_2R^2}$. So that, for $(x,t) \in \partial E_R^0(1) \times (0,\varepsilon)$ inequality $\vartheta_R(x,t) \geq u(x,t)$ is true. Now we fix $x \in \partial E_R^0(2)$ and denote through E_2^+ set $\left\{\xi : \xi \in \partial E_R^0(2), \sum\limits_{i=1}^{n}\frac{(x_i-\xi_i)^2}{R^{-\alpha_i}} \leq 4\beta E\right\}$. It is clear, that for $R \geq R_1$ $mes_{n-1}(E_2^+) \geq 1$. Then again

$$\int\limits_{\partial E_R^0(2)} \exp\left[\frac{-\sum\limits_{i=1}^{n}(x_i-\xi_i)^2/R^{-\alpha_i}}{4\beta\ E}\right] ds_\xi \geq e^{-1}$$

and that's why

$$\vartheta_R(x,t) \geq \frac{M_1 e^{C_2\lambda^2R^2}}{e(2\varepsilon)^s} = C_1 e^{C_2\lambda^2R^2}.$$

On the other hand by virtue of choice λ $E_R^0(2) \subset E_{\lambda R}^0(1)$, that involve $\sup\limits_{\partial E_R^0(2)\times(0,\varepsilon)} u(x,t) \leq C_1 e^{C_2\lambda^2R^2}$. Thus we show, that everywhere on $\Gamma(B_\varepsilon^R)$ $\vartheta_R(x,t) \geq u(x,t)$. By maximum principle this inequality is true for $(x,t) \in B_\varepsilon^R$. Now let (x',t') is an arbitrary point on $\partial E_R^0\left(\frac{3}{2}\right) \times (0,\varepsilon)$. If $\xi \in \partial E_R^0(1)$ then

$$\sqrt{\sum_{i=1}^{n}\frac{(x'-\xi_i')^2}{R^{-\alpha_i}}} \geq \sqrt{\sum_{i=1}^{n}\frac{(x_i')^2}{R^{-\alpha_i}}} - \sqrt{\sum_{i=1}^{n}\frac{\xi_i^2}{R^{-\alpha_i}}} = \frac{R}{2}.$$

But if $\xi \in \partial E_R^0(2)$, then

$$\sqrt{\sum_{i=1}^{n}\frac{(x_i'-\xi_i)^2}{R^{-\alpha_i}}} \geq \sqrt{\sum_{i=1}^{n}\frac{\xi_i^2}{R^{-\alpha_i}}} - \sqrt{\sum_{i=1}^{n}\frac{(x_i')^2}{R^{-\alpha_i}}} = \frac{R}{2}.$$

Therefore

$$u(x',t') \leq \frac{M_1 e^{C_2 R^2}}{\varepsilon^s} \int_{\partial E_R^0((1))} \exp\left[-\frac{R^2}{32\beta\ \varepsilon}\right] ds_\xi + \frac{M_1 e^{C_2 \lambda^2 R^2}}{\varepsilon^s} \int_{\partial E_R^0(2)} \exp\left[-\frac{R^2}{32\beta \varepsilon}\right] ds_\xi \leq$$

$$\leq \frac{M_1 e^{-C_2 R^2}}{\varepsilon^s} mes_{n-1}\left(\partial E_R^0(1)\right) + \frac{M_1 e^{-C_2 \lambda^2 R^2}}{\varepsilon^s} mes_{n-1}\left(\partial E_R^0(2)\right) = D(R) \ . \quad (12)$$

It is easy to see that $\lim\limits_{R\to\infty} D(R) = 0$.

So that, from (12) we deduce

$$\sup_{\partial E_R^0\left(\frac{3}{2}\right)\times(0,\varepsilon)} u(x',t') \leq D(R).$$

Now it is enough to transit in last inequality to lim for $R \to \infty$ and lemma is proved.

Corollary. *If α - slow growing function $u(x,t)$ is L- superparabolic in S_E and $u|_{t=0} \geq 0$, then*

$$\varliminf_{x\to\infty} u(x,t) \geq 0 .$$

In particular if α - slow growing function $u(x,t)$ is solution of equation of equation $Lu = 0$ in S_E and $u|_{t=0} = 0$ then $\lim\limits_{x\to\infty} u(x,t) = 0$.

Lemma 3. *Let $u(x,t)$ be solution of equation $Lu = 0$ in S_E, $u|_{t=0} = 0$. If $u(x,t) - \alpha$ - slow growing function then $u(x,t) \equiv 0$.*

Proof. We fix arbitrary $\delta > 0$ and point $(x',t') \in S_E$. According to corollary of previous lemma there exist $M > 0$ such that $|u(x,t)| \leq \delta$ for $|x| \geq M$. We denote through M_0 $\max\{M, |x'| + 1\}$ and consider cylinder $C = \{(x,t): |x| < M_0,\ 0 < t < \varepsilon\}$. It is clear that $(x',t') \in C$ and $u|_{\Gamma(C)} < \delta$. By maximum principle $u(x,t) < \delta$ for $(x,t) \in C$ and in particular

$$u(x',t') < \delta . \quad (13)$$

By the same way we prove that

$$u(x',t') > -\delta . \quad (14)$$

From (13)-(14) we obtain that $|u(x',t')| < \delta$. Because δ is arbitrary we deduce that $u(x',t') = 0$. Now it is enough to use arbitrariness of point (x',t') in S_E and lemma is proved.

Theorem. *Cauchy problem (1) has no more than one solution in class of α - slow growing functions.*

Proof. We consider layer S_E. Let $u_1(x,t)$ and $u_2(x,t)$ are two solutions of Cauchy problem (1). Then function $u(x,t) = u_1(x,t) - u_2(x,t)$ is solution of problem

$$Lu = 0\ ,\ (x,t) \in S_E\ ;\ u|_{t=0} = 0 .$$

According to Lemma 3 $u(x,t) \equiv 0$ in S_E. If $\varepsilon \geq T$ then theorem is proved. But if $\varepsilon < T$ then we consider layer $S'_E = R^+_{n+1} \cap \{(x,t): \varepsilon < t < 2\varepsilon\}$. Function $u(x,t)$ is solution of problem

$$Lu = 0\ ,\ (x,t) \in S'_E\ ;\ u|_{t=E} = 0 .$$

According to lemma 3 $u(x,t) \equiv 0$ in S'_E, i.e. $u(x,t) \equiv 0$ in S_{2E}. If $2\varepsilon \geq T$ then theorem is proved. But if $2\varepsilon < t$ then we continue process by the same way. Let m be least natural number for which $m\varepsilon \geq T$. In m steps we obtain that $u(x,t) \equiv 0$ in $S_{m\varepsilon}$. So theorem has been proved.

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Mushtagov F.M.
Institute of Mathematics and Mechanics of AS Azerbaijan.
9, F. Agayeva str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

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