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ON THE PROPERTY OF A BI-DIFFERENTIAL

Abstract

In the work the definitions of a bi-differential and two-differential are given, the classes of θ- bi-lipschitz, θ-two-lipschitz, weak θ- bi-lipschitz functions are determined. Properties of the bi-differential and two-differential are studied.

The second order sub-differentials have been considered in papers by Hiriart-Urrity J.-B. [1], Bedelbayev A.A. [2], Warga J. [3] the author [4, 5, 6] and others.

The definitions of a bi-differential (see also [7]), θ- bi-lipschitz, θ-two-lipschitz, weak θ- bi-lipschitz and θ- strong bi-lipschitz functions are given, and some of their properties are studied in the present paper.

Let X be a Banach space, $f: X \to R$. Suppose

$$f^{[1]^+}(x_0;x)=\overline{\lim_{\substack{z\to x_0\\ t\downarrow 0}}}\frac{1}{t}\left(f(z+tx)-f(z)\right),\ f^{[2]^+}(x_0;x)=\overline{\lim_{\substack{z\to x_0\\ t\downarrow 0}}}\frac{1}{t^2}\left(f(z+2tx)-2f(z+tx)+f(z)\right),$$

$$f^{[2]^-}(x_0;x)=\underline{\lim_{\substack{z\to x_0\\ t\downarrow 0}}}\frac{1}{t^2}\left(f(z+2tx)-2f(z+tx)+f(z)\right).$$

The set of all continuous bi-linear mappings from $X\times X$ into R denote by $B(X^2,R)$. If there exists such a bi-linear symmetric functional $x^*\in B(X^2,R)$, that $Q(x)=x^*(x,x)$, then $Q(x)$ given in the space X, is called a quadratic functional. The set of all quadratic functionals from X into R denote by $B_0(X)$. We call the set

$$D_2 f(x_0)=\{Q\in B_0(X): f^{[2]^-}(x_0;x)\le Q(x)\le f^{[2]^+}(x_0;x),\ \ x\in X$$

a bi-differential of the function f at the point x_0.

The following lemma immediately follows from the definition.

Lemma 1. *The following relations are valid:*

1) $f^{[2]^+}(x_0;x)=f^{[2]^+}(x_0;-x)$; *2)* $f^{[2]^-}(x_0;x)=f^{[2]^-}(x_0;-x)$;

3) $f^{[2]^+}(x_0;\alpha x)=\alpha^2 f^{[2]^+}(x_0;x)$; *4)* $f^{[2]^-}(x_0;\alpha x)=\alpha^2 f^{[2]^-}(x_0;x)$;

5) $D_2(\alpha f)(x_0)=\alpha D_2 f(x_0)$.

The function f we call 2-lipschitz with a constant K in the vicinity x_0, if for some $\varepsilon>0$ f satisfies the condition

$$|f(z+2x)-2f(z+x)+f(z)|\le K\|x\|^2 \quad (\text{or } |f(z+x)-2f(z)+f(z-x)|\le K\|x\|^2)$$

for $x\in\varepsilon B$, $z\in x_0+\varepsilon B$.

The function f we call θ-bi-lipschitz(θ-two-lipschitz) with a constant K in the vicinity x_0, if for some $\varepsilon>0$ f satisfies the condition

$$\begin{gathered}|f(z+2x)-2f(z+x)-f(z+2y)+2f(z+y)|\le K\|x-y\|^\theta(\|x\|+\|y\|)^{2-\theta},\\ (|f(z+x)+f(z-x)-f(z+y)-f(z-y)|\le K\|x-y\|^\theta(\|x\|+\|y\|)^{2-\theta})\end{gathered}\tag{1}$$

for $x,y\in\varepsilon B$, $z\in x_0+\varepsilon B$; $0<\theta<2$.

If the relation (1) is satisfied for $z = x_0$, then the function f we call θ-bi-lipschitz(θ-two-lipschitz) with constant K at the point x_0. The function f we call $\{2\}$-lipschitzwith a constant K at the point x_0, if for some $\varepsilon > 0$ f satisfies the condition

$$\left|f(x_0 + x) - 2f(x_0) + f(x_0 - x)\right| \le K\|x\|^2, \quad x \in \varepsilon B.$$

The function f we call weak θ-bi-lipschitz(θ-two-lipschitz) with a constant K at the vicinity x_0, if for some $\varepsilon > 0$ f satisfies the condition

$$\left|f(z+2x) - 2f(z+x) - f(z+2y) + 2f(z+y)\right| \le K\|x-y\|^{\theta}\left(\|x\| + \|y\|\right)^{2-\theta} + 0\left(\left(\|x\| + \|y\|\right)^2\right),$$

$$\left(\left|f(z+x) + f(z-x) - f(z+y) - f(z-y)\right| \le K\|x-y\|^{\theta}\left(\|x\| + \|y\|\right)^{2-\theta} + 0\left(\left(\|x\| + \|y\|\right)^2\right)\right)$$

for $x, y \in \varepsilon B$, $z \in x_0 + \varepsilon B$, $0 < \theta < 2$.

The function f we call θ-strong bi-lipschitz with a constant K at the point x_0, if for some $\varepsilon > 0$ f satisfies the condition

$$\left|f(x_0 + x) - f(x_0 + y)\right| \le K\|x-y\|^{\theta}\left(\|x\| + \|y\|\right)^{2-\theta}, \quad x, y \in \varepsilon B, \; 0 < \theta < 2.$$

Note that if f_i, $i = \overline{1,n}$ satisfies the θ-strong bi-lipschitz condition at the point x_0, then $\max\limits_{i=1,n} f_i(x)$ also satisfies θ-strong bi-lipschitz condition at the point x_0.

It is clear that if f is a θ-bi-lipschitz (θ-two-lipschitz) function in the vicinity x_0, then f is $\{2\}$-lipschitz the vicinity x_0.

Let $\alpha > 0$, $\nu > 0$, $\beta \ge \alpha\nu$ and $\delta > 0$. The function f we call a $(\alpha, \beta, \nu, \delta)$ lipschitz (see [5]) with a constant K at the point x_0, if f satisfies the condition

$$\left|f(x_0 + x + y) - f(x_0 + x)\right| \le K\|y\|^{\nu}\left(\|x\|^{\beta - \alpha\nu} + \|y\|^{\frac{\beta - \alpha\nu}{\alpha}}\right), \quad x, y \in \delta B.$$

We can easily check that if the function f satisfies the $(1, 2, \nu, \delta)$ lipschitz condition with a constant K at the point x_0, then

$$\left|f(x_0 + x) - f(x_0 + y)\right| \le K\|x-y\|^{\nu}\left(\|x\|^{2-\nu} + \|x-y\|^{2-\nu}\right).$$

Lemma 2. *If the function f_τ, $\tau \in \Omega$ satisfies the $(\alpha, \beta, \nu, \delta)$ lipschitz condition with a constant L_τ at the point x_0 and $L = \sup\{L_\tau : \tau \in \Omega\} < +\infty$, then $f(x) = \sup\limits_{\tau \in \Omega} f_\tau(x)$ also satisfies the $(\alpha, \beta, \nu, \delta)$- lipschitz condition with a constant L at the point x_0.*

Proof. It is clear that for $x, y \in \delta B$

$$f(x_0 + x + y) - f(x_0 + x) = \sup_{\tau \in \Omega} f_\tau(x_0 + x + y) - \sup_{\tau \in \Omega} f_\tau(x_0 + x) \le$$

$$\le \sup_{\tau \in \Omega}\left(f_\tau(x_0 + x + y) - f_\tau(x_0 + x)\right) \le L\|y\|^{\nu}\left(\|x\|^{\beta - \alpha\nu} + \|y\|^{\frac{\beta - \alpha\nu}{\alpha}}\right),$$

$$f(x_0 + x + y) - f(x_0 + x) = \sup_{\tau \in \Omega} f_\tau(x_0 + x + y) + \inf_{\tau \in \Omega}\left(-f_\tau(x_0 + x)\right) \ge \tag{2}$$

$$\ge \inf_{\tau \in \Omega}\left(f_\tau(x_0 + x + y) - f_\tau(x_0 + x)\right) \ge -L\|y\|^{\nu}\left(\|x\|^{\beta - \alpha\nu} + \|y\|^{\frac{\beta - \alpha\nu}{\alpha}}\right).$$

It follows from the relation (2) that f satisfies the $(\alpha, \beta, \nu, \delta)$- lipschitzcondition with a constant L at the point x_0. The proof of lemma is completed.

Theorem 1. *If f is a θ-bi- lipschitz function with a constant K in the vicinity u, then $f^{|2|^*}(u;\upsilon)$ is an upper semicontinuous function depending on $(u;\upsilon)$, and as a function only on υ satisfies the condition*

$$\left|f^{|2|^*}(u;x)-f^{|2|^*}(u;\upsilon)\right|\le K\|x-\upsilon\|^\theta(\|x\|+\|\upsilon\|)^{2-\theta}.$$

Proof. Let $\{y_i$ and $\{\upsilon_i\}$ be arbitrary sequences converging respectively to u and υ. For each i by definition of upper limit there exist such $z_i\in X$ and $t_i>0$ that $\|z_i-y_i\|+t_i<\frac{1}{i}$ and

$$f^{|2|^*}(y_i;\upsilon_i)-\frac{1}{i}\le\frac{f(z_i+2t_i\upsilon_i)-2f(z_i+t_i\upsilon_i)+f(z_i)}{t_i^2}=\frac{f(z_i+2t_i\upsilon)-2f(z_i+t_i\upsilon)+f(z_i)}{t_i^2}+$$

$$+\frac{f(z_i+2t_i\upsilon_i)-f(z_i+2t_i\upsilon)+2f(z_i+t_i\upsilon)-2f(z_i+t_i\upsilon_i)}{t_i^2}\le$$

$$\le\frac{f(z_i+2t_i\upsilon)-2f(z_i+t_i\upsilon)+f(z_i)}{t_i^2}+K\|\upsilon_i-\upsilon\|^\theta(\|\upsilon_i\|+\|\upsilon\|)^{2-\theta}.$$

Passing to upper limits for $i\to\infty$ we get

$$\overline{\lim} f^{|2|+}(y_i;\upsilon_i)\le f^{|2|+}(y;\upsilon),$$

i.e. $f^{|2|+}(u;\upsilon)$ is an upper semocontinuous function depending on (u,υ).

If $x,\upsilon\in X$, then

$$f(y+2\lambda x)-2f(y+\lambda x)+f(y)\le f(y+2\lambda\upsilon)-2f(y+\lambda\upsilon)+f(y)+$$
$$+K\lambda^2\|x-\upsilon\|^\theta(\|x\|+\|\upsilon\|)^{2-\theta}$$

for y near u, λ near 0. Division by λ and passage to upper limits for $y\to u$, $\lambda\downarrow 0$ gives the inequality

$$f^{|2|+}(u,x)\le f^{|2|+}(u,\upsilon)+K\|x-\upsilon\|^\theta(\|x\|+\|\upsilon\|)^{2-\theta}. \quad (3)$$

In this inequality by replacing x and υ we get

$$f^{|2|+}(u;\upsilon)\le f^{|2|+}(u;x)+K\|x-\upsilon\|^\theta(\|x\|+\|\upsilon\|)^{2-\theta}. \quad (4)$$

We get (3) and (4)

$$\left|f^{|2|+}(u;x)-f^{|2|+}(u;\upsilon)\right|\le K\|x-\upsilon\|^\theta(\|x\|+\|\upsilon\|)^{2-\theta}.$$

The theorem is proved.

The following theorem is proved analogously.

Theorem 2. *If f is a θ-bi- lipschitz function with a constant K in the vicinity u, then $f^{|2|-}(u,\upsilon)$ is an lower semicontinuous function depending on $(u;\upsilon)$ and as the one-variable function only by the second argument satisfies the condition*

$$\left|f^{|2|-}(u;x)-f^{|2|-}(u;\upsilon)\right|\le K\|x-\upsilon\|^\theta(\|x\|+\|\upsilon\|)^{2-\theta}$$

for any $x,\upsilon\in X$.

The following corollary arises from theorem 1 and 2.

Corollary 1. *If X is finite dimensional and f θ-bi- lipschitzfunction in the vicinity x_0, then $D_2f(x)$ is upper semicontinuous in x_0.*

Assume

$$D_2^+ f(x_0) = \{Q \in B_0(X): f^{|2|+}(x_0;x) \geq Q(x),\ x \in X\},$$
$$D_2^- f(x_0) = \{Q \in B_0(X): f^{|2|-}(x_0;x) \leq Q(x),\ x \in X\}.$$

It is clear that $D_2 f(x_0) = D_2^+ f(x_0) \cap D_2^- f(x_0)$. Besides, if $D_2 f(x_0)$ is not empty, then

$$f^{|2|-}(x_0;x) \leq \inf_{Q \in D_2 f(x_0)} Q(x) \leq \sup_{Q \in D_2 f(x_0)} Q(x) \leq f^{|2|+}(x_0;x).$$

Assume

$$f^{(2)+}(x_0;x) = \overline{\lim_{t \downarrow 0}} \frac{1}{t^2}\left(f(x_0+tx) - 2f(x_0) + f(x_0-tx)\right),$$

$$f^{(2)-}(x_0;x) = \underline{\lim_{t \downarrow 0}} \frac{1}{t^2}\left(f(x_0+tx) - 2f(x_0) + f(x_0-tx)\right).$$

If f satisfies the θ-two- lipschitzcondition with a constant K at the point x_0, then

$$\left|f^{(2)+}(x_0;x) - f^{(2)+}(x_0;y)\right| \leq K\|x-y\|^\theta \left(\|x\| + \|y\|\right)^{2-\theta},$$
$$\left|f^{(2)-}(x_0;x) - f^{(2)-}(x_0;y)\right| \leq K\|x-y\|^\theta \left(\|x\| + \|y\|\right)^{2-\theta}.$$

The set $\check{D}_2 f(x_0) = \{Q \in B_0(X): f^{(2)-}(x_0;x) \leq Q(x) \leq f^{(2)+}(x_0;x),\ x \in X\}$ we call a 2-differential of the function f at the point x_0.

Assume $\check{D}_2^+ f(x_0) = \{Q \in B_0(X): f^{(2)+}(x_0;x) \geq Q(x),\ x \in X\}$,
$\check{D}_2^- f(x_0) = \{Q \in B_0(X): f^{(2)-}(x_0;x) \leq Q(x),\ x \in X\}$.

It is clear that $\check{D}_2 f(x_0) = \check{D}_2^+ f(x_0) \cap \check{D}_2^- f(x_0)$. Let $\overline{B}(X^2,R) = \{x^* \in B(X^2,R): x^* \text{ is symmetric}\}$.

Lemma 3. *If g is a positive homogeneous second degree non-negative continuous function from X to R and $g(-x) = g(x)$, then*

$$g(x) = \max\{Q(x): Q \in G\},$$

where $G = \check{D}_2^+\left(\frac{1}{2}g\right)(0) = \{Q \in B_0(X): g(x) \geq Q(x), x \in X\}$.

Proof. Assume that $P(x,y) = \sqrt{g(x)g(y)}$. It is clear that P is a bipositve homogeneous continuous function from $X \times X$ to R and $P(-x,-y) = P(x,y)$. Therefore by analogy lemma 4 and theorem 1 [7] we obtain

$$P(x,x) = \max\{x^*(x,x): x^* \in \partial_2 P\},$$

where $\partial_2 P = \{x^* \in \overline{B}(X^2,R): P(x,y) \geq x^*(x,y),\ x,y \in X\}$. Therefore we get

$$g(x) = P(x,x) = \max\{x^*(x,x): x^* \in \partial_2 P\} = \max\{Q(x): Q \in G\}.$$

The lemma is proved.

Lemma 4. *If f is a θ-two-lipschitz function with a constant K at x_0, then* $\check{D}_2^+\left(f + \frac{1}{2}K\|\cdot - x_0\|^2\right)(x_0)$ *and* $\check{D}_2^+\left(\frac{1}{2}K\|\cdot - x_0\|^2 - f\right)(x_0)$ *are not empty and*

$$f^{(2)+}(x_0;x) = \max\left\{Q(x) - K\|x\|^2 : Q \in \check{D}_2^+\left(f + \frac{1}{2}K\|\cdot - x_0\|^2\right)(x_0)\right\},$$

$$f^{(2)-}(x_0;x)=\min\left\{K\|x\|^2-Q(x):Q\in \check{D}_2^+\left(\frac{1}{2}K\|\cdot-x_0\|^2-f\right)(x_0)\right\}.$$

Proof. Denote $g_1(x)=f(x)+\frac{1}{2}K\|x-x_0\|^2$, $g_2(x)=\frac{1}{2}K\|x-x_0\|^2-f(x)$. It is clear that

$$g_1^{(2)+}(x_0;x)=\overline{\lim_{t\downarrow 0}}\frac{1}{t^2}\left(f(x_0+tx)-2f(x_0)+f(x_0-tx)+Kt^2\|x\|^2\right)=f^{(2)+}(x_0;x)+K\|x\|^2,$$

$$g_2^{(2)+}(x_0;x)=\overline{\lim_{t\downarrow 0}}\frac{1}{t^2}\left(-f(x_0+tx)+2f(x_0)-f(x_0-tx)+Kt^2\|x\|^2\right)=K\|x\|^2-f^{(2)-}(x_0;x).$$

According to condition f is a $\{2\}$- lipschitzfunction with a constant K at the point x_0, therefore $g_1^{(2)+}(x_0;x)\geq 0$, $g_2^{(2)+}(x_0;x)\geq 0$. Besides, by lemma 3 we get

$$g_1^{(2)+}(x_0;x)=\max\left\{Q(x):Q\in \check{D}_2^+\left(f+\frac{1}{2}K\|\cdot-x_0\|^2\right)(x_0)\right\},$$

$$g_2^{(2)+}(x_0;x)=\max\left\{Q(x):Q\in \check{D}_2^+\left(\frac{1}{2}K\|\cdot-x_0\|^2-f\right)(x_0)\right\}.$$

Since $g_1^{(2)+}(x_0;x)=f^{(2)+}(x_0;x)+K\|x\|^2$, $g_2^{(2)+}(x_0;x)=K\|x\|^2-f^{(2)-}(x_0;x)$, then we get

$$f^{(2)+}(x_0;x)=\max\left\{Q(x)-K\|x\|^2:Q\in \check{D}_2^+\left(f+\frac{1}{2}K\|\cdot-x_0\|^2\right)(x_0)\right\},$$

$$f^{(2)-}(x_0;x)=\min\left\{K\|x\|^2-Q(x):Q\in \check{D}_2^+\left(\frac{1}{2}K\|\cdot-x_0\|^2-f\right)(x_0)\right\}.$$

The lemma is proved.

Corollary 2. *If X is a Hilbert space and f is a two-lipschitz function at the point x_0, then $\check{D}_2^+f(x_0)$ and $\check{D}_2^-f(x_0)$ are not empty and*

$$f^{(2)+}(x_0;x)=\max\left\{Q(x):Q\in \check{D}_2^+f(x_0)\right\},$$

$$f^{(2)-}(x_0;x)=\min\left\{Q(x):Q\in \check{D}_2^-f(x_0)\right\}.$$

Proof. It is clear that in a Hilbert space $\|x\|^2=\langle x,x\rangle$ is a quadratic functional. Therefore

$$\check{D}_2^+\left(f+\frac{1}{2}K\|\cdot-x_0\|^2\right)(x_0)=\left\{Q\in B_0(X):f^{(2)+}(x_0;x)\geq Q(x)-K\|x\|^2,\ x\in X\right\}=$$

$$=\check{D}_2^+f(x_0)+K\|x\|^2,$$

$$\check{D}_2^+\left(\frac{1}{2}K\|\cdot-x_0\|^2-f\right)(x_0)=\left\{Q\in B_0(X):f^{(2)-}(x_0;x)\leq K\|x\|^2-Q(x),\ x\in X\right\}=$$

$$=K\|x\|^2-\check{D}_2^-f(x_0).$$

Considering these relations in lemma 4, we get

$$f^{(2)+}(x_0,x)=\max\left\{Q(x)-K\|x\|^2:Q\in \check{D}_2^+f(x_0)+K\|x\|^2\right\}=\max\left\{Q(x):Q\in \check{D}_2^+f(x_0)\right\},$$

$$f^{(2)-}(x_0,x)=\min\left\{K\|x\|^2-Q(x):Q\in K\|x\|^2-\check{D}_2^- f(x_0)\right\}=\min\left\{Q(x):Q\in \check{D}_2^- f(x_0)\right\}.$$

The corollary is proved.

Denoting $g_1(x)=f^{[2]+}(x_0;x)+K\|x\|^2$, $g_2(x)=K\|x\|^2-f^{[2]-}(x_0;x)$ from lemma 3 we get that the following corollary 3 is valid.

Corollary 3. *If X is a Hilbert space and f is a θ-bi- lipschitzfunction in the vicinity x_0, then $D_2^+ f(x_0)$ and $D_2^- f(x_0)$ are not empty and*

$$f^{|2|+}(x_0;x)=\max\{Q(x):Q\in D_2^+ f(x_0)\},$$
$$f^{|2|-}(x_0;x)=\min\{Q(x):Q\in D_2^- f(x_0)\}.$$

Let $\psi_i : X\times X \to R$ be bipositive homogeneous symmetric functions. The function f we call the (θ,ψ_1,ψ_2)-bi- lipschitzin the vicinity x_0, if for some $\varepsilon>0$ the function f satisfies the condition

$$|f(z+x_1+x_2)-f(z+x_1)-f(z+x_2)-f(z+y_1+y_2)+f(z+y_1)+f(z+y_2)|\le$$
$$\le|\psi_1(x_1,x_2)-\psi_1(y_1,y_2)|^{\frac{\theta}{2}}\cdot|\psi_2(x_1,x_2)+\psi_2(y_1,y_2)|^{\frac{2-\theta}{2}}+0(\|x_1\|\cdot\|x_2\|+\|y_1\|\cdot\|y_2\|),$$

where $x_1,x_2,y_1,y_2\in\varepsilon B$, $z\in x_0+\varepsilon B$, $0<\theta\le 2$, $\frac{o(\lambda)}{\lambda}\to 0$ for $\lambda\downarrow 0$.

Lemma 5. *If f satisfies the (θ,ψ_1,ψ_2)-bi- lipschitzcondition in the vicinity x_0, then $d_2 f(x_0)=D_2 f(x_0)$, where $d_2 f(x_0)$ is defined in [7].*

Proof. It is clear that (see [7])

$$f^{[2]}(x_0;x,x)-f^{|2|+}(x_0;x)\le \varlimsup_{\substack{z\to x_0\\ \lambda_1\downarrow 0,\lambda\downarrow 0}}\frac{1}{\lambda^2}\left[f\left(z+\lambda_1 x+\frac{\lambda^2}{\lambda_1}x\right)-f(z+\lambda_1 x)-\right.$$
$$\left.-f\left(z+\frac{\lambda^2}{\lambda}x\right)-f(z+2\lambda x)+2f(z+\lambda x)\right]\le \varlimsup_{\substack{z\to x_0\\ \lambda_1\downarrow 0,\lambda\downarrow 0}}\frac{1}{\lambda^2}o\left(2\lambda^2\|x\|^2\right)=0.$$

Similarly we obtain $f^{|2|+}(x_0;x)-f^{|2|}(x_0;x,x)\le 0$. Therefore $f^{[2]+}(x_0;x)=$ $=f^{[2]}(x_0;x,x)$.

Analogously it is verified that $f^{|2|-}(x_0;x)=f_{|2|}(x_0;x,x)$. From relations $f^{|2|+}(x_0;x)=f^{|2|}(x_0;x,x)$ and $f^{|2|-}(x_0;x)=f_{|2|}(x_0;x,x)$ follows $d_2 f(x_0)=D_2 f(x_0)$. The lemma is proved.

Note that in definition of the (θ,ψ_1,ψ_2)-bi- lipschitz function it is appropriate to assume $y_1=y_2$ and $\psi_2(x_1,x_2)=\|x_1\|\cdot\|x_2\|$.

Corollary 4. *If f is a (θ,ψ_1,ψ_2)-bi- lipschitzfunction in the vicinity x_0, then $D_2 f(x_0)$ is not empty and*

$$f^{|2|+}(x_0;x)=\max\{Q(x):Q\in D_2 f(x_0)\},$$
$$f^{|2|-}(x_0;x)=\min\{Q(x):Q\in D_2 f(x_0)\}.$$

Corollary 4 immediately follows from corollary 3[7].

If f is convex, $\alpha_1\ge 0$, $\alpha_2\ge 0$, $\alpha_1+\alpha_2=1$, then

$$f^{[2]+}(x_0;\alpha_1 x_1+\alpha_2 x_2)=\overline{\lim_{\substack{z\to x_0\\ t\downarrow 0}}}\frac{f(z+t(\alpha_1 x_1+\alpha_2 x_2))-2f(z)+f(z-t(\alpha_1 x_1+\alpha_2 x_2))}{t^2}\le$$

$$\le\alpha_1\overline{\lim_{\substack{z\to x_0\\ t\downarrow 0}}}\frac{1}{t^2}(f(z+tx_1)-2f(z)+f(z-tx_1))+\alpha_2\overline{\lim_{\substack{z\to x_0\\ t\downarrow 0}}}\frac{1}{t^2}(f(z+tx_2)-2f(z)+f(z-tx_2))=$$

$$=\alpha_1 f^{|2|+}(x_0;x_1)+\alpha_2 f^{|2|+}(x_0;x_2),$$

i.e. $x\to f^{|2|+}(x_0;x)$ is convex. Besides

$$f^{|2|-}(x_0;x)=\underline{\lim_{\substack{z\to x_0\\ t\downarrow 0}}}\frac{1}{t^2}\left(2\left(\frac{1}{2}f(z+2tx)+\frac{1}{2}f(z)\right)-2f(z+tx)\right)\ge 0.$$

It is analogously verified that f is convex, then $x\to f^{(2)+}(x_0;x)$ is convex and $f^{(2)+}(x_0;x)\ge 0$.

Let f be convex. The set $D_2^{\oplus}f(x_0)=\left\{x^*\in\overline{B}(X^2,R):\sqrt{f^{(2)+}(x_0;x)\cdot f^{(2)+}(x_0;y)}\ge\right.$ $\left.\ge x^*(x,y),\ x,y\in X\right\}$ we call the $\{2\}$-subdifferential of the function f at the point x_0.

Let X be a Hilbert space and the function f satisfy the $\{2\}$- lipschitzcondition with a constant K at the point x_0. The set

$$\overset{\vee}{D}{}^2 f(x_0)=\left\{x^*\in\overline{B}(X^2,R):\sqrt{\left(f^{(2)+}(x_0;x)+K\cdot\|x\|^2\right)\cdot\left(f^{(2)+}(x_0;y)+K\cdot\|y\|^2\right)}\ge\right.$$

$$\ge x^*(x,y)+K<x,y>;x,y\in X$$

we call the $\{2\}$-differential of the function f at the point x_0. It is clear that

$$f^{(2)+}(x_0;x)=\max\left\{x^*(x,x):x^*\in\overset{\vee}{D}{}^2 f(x_0)\right\}.$$

They say that (see [8]), f has a strict derivative $D_s f(x_0)$ at the point x_0, if

$$\lim_{\substack{z\to x_0\\ t\downarrow 0}}\frac{f(z+tx)-f(z)}{t}=<D_s f(x_0),x>,$$

where the convergence is uniform with regard to x on any compact set.

Theorem 3. *Let X be a Hilbert space, f, g and ψ be (θ,ψ_1,ψ_2)- bi-lipschitzfunctions in the vicinity x_0 and f has a strict derivate $D_s f(x_0)$ at the point x_0. Then*

1) $D_2(g+\psi)(x_0)\subset D_2 g(x_0)+D_2\psi(x_0)$,

2) $D_2(fg)(x_0)\subset g(x_0)D_2 f(x_0)+f(x_0)D_2 g(x_0)+2D_s f(x_0)\otimes\partial g(x_0)$,

where $D_s f(x_0)\otimes\partial g(x_0)=\{<D_s f(x_0),x>\cdot x^*(x):x^*\in\partial g(x_0)\}$.

Proof. Relation 1) follows from Corollary 4 and definition of the bidifferential. Prove relation 2). It is clear that

$$(f.g)^{[2]+}(x_0;x)=\overline{\lim_{\substack{z\to x_0\\ \lambda\downarrow 0}}}\frac{f(z+\lambda x)g(z+\lambda x)-2f(z)g(z)+f(z-\lambda x)g(z-\lambda x)}{\lambda^2}\le$$

$$\leq \overline{\lim_{\substack{z\to x_0\\ \lambda\downarrow 0}}} \frac{g(z)(f(z+\lambda x)-2f(z)+f(z-\lambda x))}{\lambda^2}+$$

$$+\overline{\lim_{\substack{z\to x_0\\ \lambda\downarrow 0}}} \frac{f(z+\lambda x)(g(z+\lambda x)-2g(z)+g(z-\lambda x))}{\lambda^2}+$$

$$+\overline{\lim_{\substack{z\to x_0\\ \lambda\downarrow 0}}} \frac{(f(z+\lambda x)-f(z-\lambda x))(g(z)-g(z-\lambda x))}{\lambda^2}.$$

Under the condition of Theorem 3 we have

$$(f.g)^{[2]+}(x_0,x)\leq \begin{cases} g(x_0)f^{[2]+}(x_0;x); g(x_0)\geq 0 \\ g(x_0)f^{[2]-}(x_0;x); g(x_0)<0 \end{cases} + \begin{cases} f(x_0)g^{[2]+}(x_0;x); f(x_0)\geq 0 \\ f(x_0)g^{[2]-}(x_0;x); f(x_0)<0 \end{cases}+$$

$$+\begin{cases} 2<D_s f(x_0),x>\cdot g^{[1]+}(x_0;x);\ <D_s f(x_0),x>\geq 0 \\ 2<D_s f(x_0),x>\cdot g^{[1]-}(x_0;x);\ <D_s f(x_0),x><0 \end{cases}.$$

Analogously we have

$$(f.g)^{[2]-}(x_0,x)\leq \begin{cases} g(x_0)f^{[2]-}(x_0;x); g(x_0)\geq 0 \\ g(x_0)f^{[2]+}(x_0;x); g(x_0)<0 \end{cases} + \begin{cases} f(x_0)g^{[2]-}(x_0;x); f(x_0)\geq 0 \\ f(x_0)g^{[2]+}(x_0;x); f(x_0)<0 \end{cases}+$$

$$+\begin{cases} 2<D_s f(x_0),x>\cdot g^{[1]-}(x_0;x);\ <D_s f(x_0),x>\geq 0 \\ 2<D_s f(x_0),x>\cdot g^{[1]+}(x_0;x);\ <D_s f(x_0),x><0 \end{cases}.$$

Therefore we get from the definition of the bidiferential and Corollary 4 that

$$D_2(f.g)(x_0)\subset g(x_0)D_2 f(x_0)+f(x_0)D_2 g(x_0)+2D_s f(x_0)\otimes \partial g(x_0).$$

The theorem is proved.

We say that f has a strict second derivative $Q(x)$ $(Q(x)\in B_0(X))$ at the point x_0, if

$$\lim_{\substack{z\to x_0\\ \lambda\downarrow 0}} \frac{f(z+2\lambda x)-2f(z+\lambda x)+f(z)}{\lambda^2}=Q(x),$$

where the convergence is uniform with respect to x on any compact set.

Lemma 6. *If f has a strict second derivative $Q(x)$ at the point x_0, then f is a {2}- lipschitz function in the vicinity x_0 and for any $x\in X$*

$$\lim_{\substack{z\to x_0\\ \lambda\downarrow 0}} \frac{1}{\lambda^2}(f(z+2\lambda x)-2f(z+\lambda x)+f(z))=Q(x).$$

Proof. If f has a strict second derivative, then it is obvious that the equality is satisfied. Show that f is a $\{2\}$- lipschitz function in the vicinity x_0. If it is not the case, then there exist the sequences $\{z_i$ and $\{x_i$ converging to x_0 and 0, such that $z_i\in x_0+$ $+\frac{1}{i}B$, $x_i\in\frac{1}{i}B$ and

$$|f(z_i+2x_i)-2f(z_i+x_i)+f(z_i)|\geq i\|x_i\|^2 . \tag{5}$$

Define t_i and υ_i by the following way: $x_i=t_i\upsilon_i$, $\|\upsilon_i\|=\frac{1}{\sqrt[3]{i}}$. It is obvious that $t_i\to 0$. It is clear that $V=\{0,\upsilon_i; i=1,2,...$ is a compactum. Therefore, by definition of the strict second derivative for any $\varepsilon > 0$ one can find such a number n_ε, that for all $i \geq n_\varepsilon$ and all $\upsilon \in V$

$$\left| \frac{f(z_i + 2t_i \upsilon) - 2f(z_i + t_i \upsilon) + f(z_i)}{t_i^2} - Q(\upsilon) \right| < \varepsilon .$$

But it is impossible, since when $\upsilon = \upsilon_i$ we get from (5) that

$$\left| \frac{f(z_i + 2t_i \upsilon_i) - 2f(z_i + t_i \upsilon_i) + f(z_i)}{t_i^2} \right| \geq i \|\upsilon_i\|^2 = \sqrt[3]{i} .$$

The lemma is proved.

Lemma 7. *If is a f θ-bi- lipschitz function in the vicinity x_0 and*

$$\lim_{\substack{z \to x_0 \\ \lambda \downarrow 0}} \frac{1}{\lambda^2} \left(f(z + 2\lambda x) - 2f(z + \lambda x) + f(z) \right) = Q(x) \tag{6}$$

for any $x \in X$, then f has a strict second derivative $Q(x)$ at the point x_0.

Proof. Let V be a compact set in X and $\varepsilon > 0$. By relation (6) for each $x \in V$ there exists such a number $\delta(x) > 0$, that

$$\left| \frac{f(z + 2tx) - 2f(z + tx) + f(z)}{t^2} - Q(x) \right| < \varepsilon \tag{7}$$

for all $z \in x_0 + \delta B$ and $t \in (0, \delta(x))$. Since

$$\left| \frac{f(z + 2t\upsilon) - 2f(z + t\upsilon) + f(z)}{t^2} - \frac{f(z + 2tx) - 2f(z + tx) + f(z)}{t^2} \right| < K \|\upsilon - x\|^\theta \left(\|\upsilon\| + \|x\| \right)^{2-\theta}$$

then appropriately over-determining $\delta(\upsilon) = \delta$ from (7) we get

$$\left| \frac{f(z + 2t\upsilon) - 2f(z + t\upsilon) + f(z)}{t^2} - Q(\upsilon) \right| < 2\varepsilon \tag{8}$$

for all $z \in x_0 + \delta B$, $\upsilon \in x + \delta B$ and $t \in (0, \delta)$. There exists a finite range of the form $\{\upsilon + \delta(\upsilon)B : \upsilon \in V\}$ of the set V determined by the vectors $\upsilon_1, \ldots, \upsilon_n$. If we assume $\delta' = \min_{1 \leq i \leq n} \delta(\upsilon_i)$ we get that (8) is fulfilled for any $z \in x_0 + \delta\mathcal{B}$, $\upsilon \in V$ and $t \in (0, \delta')$, i.e. $Q(x)$ is the strict second derivative. The lemma is proved.

It is clear that, if $(x, \alpha) \in ep\, f^{[2]+}(x_0; .) = \{(x, \alpha) \in X \times R : f^{[2]+}(x_0; x) \leq \alpha$ and $\beta \geq 0$, then $(\beta x, \beta^2 \alpha) \in ep\, f^{[2]+}(x_0; .)$, $(-\beta x, \beta^2 \alpha) \in ep\, f^{[2]+}(x_0; .)$.

Let $C \subset X$. Assume $d(y) = \inf\{\|y - z\| : z \in C\}$, $d_2(y) = d^2(y)$.

Lemma 8. *If C is a non-empty convex subset of the Euclidean space Y, then at any z, $x, y \in Y$ the relation*

$$\left| d_2(z + 2x) - d_2(z + 2\upsilon) - 2d_2(z + x) + 2d_2(z + \upsilon) \right| \leq 10 \|x - \upsilon\| \left(\|x\| + \|\upsilon\| \right)$$

is fulfilled.

Proof. Let $c_1, c_2 \in C$ be such that $d(z + 2\upsilon) = \|z + 2\upsilon - c_1\|$, $d(z + x) = \|z + x - c_2\|$. By using theorem 3.4.8 [9] we get

$$d_2(z + 2x) - d_2(z + 2\upsilon) - 2d_2(z + x) + 2d_2(z + \upsilon) \leq \|z + 2x - c_1\|^2 - \|z + 2\upsilon - c_1\|^2 -$$

$$- 2\|z + x - c_2\|^2 + 2\|z + \upsilon - c_2\|^2 = \|z + x + \upsilon - c_1 + (x - \upsilon)\|^2 - \|z + x + \upsilon - c_1 - (x - \upsilon)\|^2 +$$

$$+\frac{1}{2}\left(\|2z+x+\upsilon-2c_2+(\upsilon-x)\|^2-\|2z+x+\upsilon-2c_2-(\upsilon-x)\|^2\right)=$$
$$=<x-\upsilon,4z+4x+4\upsilon-4c_1>-<x-\upsilon,4z+2x+2\upsilon-4c_2>=$$
$$=<x-\upsilon,2x+2\upsilon-4(c_1-c_2)>.$$

It is easy verified that if C is convex, then $\|c_1-c_2\|\le\|2\upsilon-x\|$. Therefore

$$d_2(z+2x)-d_2(z+2\upsilon)-2d_2(z+x)+2d_2(z+\upsilon)\le\|x-\upsilon\|(2\|x+\upsilon\|+4\|2\upsilon-x\|)\le$$
$$\le\|x-\upsilon\|(2\|x\|+2\|\upsilon\|+8\|\upsilon\|+4\|x\|)\le 10\|x-\upsilon\|(\|x\|+\|\upsilon\|). \quad (9)$$

Analogously it is verified that

$$d_2(z+2x)-d_2(z+2\upsilon)-2d_2(z+x)+2d_2(z+\upsilon)\ge-10\|x-\upsilon\|(\|x\|+\|\upsilon\|). \quad (10)$$

It follows from (9) and (10) that the lemma is valid.

If $z\in C$, then it is easily verified that

$$|d_2(z+2x)-d_2(z+2\upsilon)-2d_2(z+x)+2d_2(z+\upsilon)|\le 6\|x-\upsilon\|(\|x\|+\|\upsilon\|),$$
$$|d_2(z+x)+d_2(z-x)-d_2(z+\upsilon)-d_2(z-\upsilon)|\le 2\|x-\upsilon\|(\|x\|+\|\upsilon\|).$$

The vector x is called an admissible direction (see [10]) of the set C at the point x_0, if one can find such a number $\lambda_x>0$, that $x_0+\lambda x\in C$ for any $\lambda\in(0,\lambda_x)$. The totality of all admissible directions at the point x_0 of the set C denote by $\gamma(x_0;C)$. Assume $uep\ f=\{(x,\alpha)\in X\times R_+: f(x)\le\alpha^2\}$, $lep\ f=\{(x,\beta)\in X\times R_+: f(x)\le-\beta^2\}$.

It is easily verified that the following lemma is valid.

Lemma 9. *If* $f\ge0$, $\pm(y,\alpha)\in\gamma\left(\left(x_0,\sqrt{f(x_0)}\right);uep\ f\right)$, *then* $f^{(2)+}(x_0;y)\le2\alpha^2$.

Assume $g(x)=f(x_0+x)-2f(x_0)+f(x_0-x)$. It is clear that

$$f^{(2)+}(x_0;x)=\overline{\lim_{\lambda\downarrow0}}\frac{g(\lambda x)}{\lambda^2},\quad f^{(2)-}(x_0;x)=\underline{\lim_{\lambda\downarrow0}}\frac{g(\lambda x)}{\lambda^2}.$$

Lemma 10. *If* $f^{(2)+}(x_0;x)\le\alpha^2$, $\alpha\ge0$, *then there exists* $\eta>0$ *and* $0(\lambda):[0,\eta]\to R_+$, *where* $\frac{0(\lambda)}{\lambda}\to0$ *for* $\lambda\downarrow0$, *that* $(\lambda x,\lambda\alpha+0(\lambda))\in uep g$ *for* $\lambda\in[0,\eta]$.

Proof. Since $\overline{\lim_{\lambda\downarrow0}}\frac{g(\lambda x)}{\lambda^2}\le\alpha^2$, then $\inf_{\eta>0}\sup_{0<\lambda<\eta}\frac{g(\lambda x)}{\lambda^2}\le\alpha^2$. Therefore for any $\varepsilon>0$ there exists $\eta_\varepsilon>0$, that $\frac{g(\lambda x)}{\lambda^2}\le\alpha^2+\varepsilon$ for $0<\lambda\le\eta_\varepsilon$. If $\varepsilon=\frac{1}{k}$, then there exists $\eta_k>0$, that $\frac{g(\lambda x)}{\lambda^2}\le\alpha^2+\frac{1}{k}$ for $0<\lambda\le\eta_k$. Assuming $0_1(\lambda)=\frac{1}{k}$ for $\lambda\in(\eta_{k+1},\eta_k)$ we get that $0_1(\lambda)\to0$ for $\lambda\downarrow0$ and $g(\lambda x)\le\alpha^2\lambda^2+\lambda^2 0_1(\lambda)$ for $\lambda\in(0,\eta_1]$. If $0(\lambda)=\lambda\sqrt{0_1(\lambda)}$, then we get that $g(\lambda x)\le(\lambda\alpha+0(\lambda))^2$ for $\lambda\in(0,\eta_1]$, i.e. $(\lambda x,\lambda\alpha+0(\lambda))\in uep\ g$ for $\lambda\in(0,\eta_1]$. The lemma is proved.

The following lemma is proved similar to lemma 10.

Lemma 11. $f^{(2)-}(x_0,x)\ge-\beta^2$, $\beta\ge0$, *if and only there exist* $\eta>0$ *and* $0(\lambda):[0,\eta]\to R_+$, *where* $\frac{0(\lambda)}{\lambda}\to0$ *for* $\lambda\downarrow0$, *that* $(\lambda x,\lambda\beta+0(\lambda))\in lep\ g$ *for* $\lambda\in[0,\eta]$.

Let $\psi:X\to R$, $\psi(0)=0$. Assume

$$\gamma^+((0,0);uep\psi)=\{(x,\alpha)\in X\times R_+:(\lambda x,\lambda\alpha+0(\lambda))\in uep\ \psi$$

$$\text{for } \lambda\in(0,\eta_{x,\alpha}),\eta_{x,\alpha}>0\ ,\ \frac{0(\lambda)}{\lambda}\to 0 \quad \text{for} \quad \lambda\downarrow 0\ ,$$

$$\gamma^-((0,0);lep\psi)=\{(x,\beta)\in X\times R_+:(\lambda x,\lambda\beta+0(\lambda))\in lep\ \psi$$

$$\text{for } \lambda\in(0,\eta_{x,\beta}),\eta_{x,\beta}>0\ ,\ \frac{0(\lambda)}{\lambda}\to 0 \quad \text{for} \quad \lambda\downarrow 0\ ,$$

$$N^{(2)+}(\psi(0))=\{Q\in B_0(X):Q(x)-\alpha^2\le 0 \text{ for } (x,\alpha)\in\gamma^+((0,0);uep\ \psi)\ ,$$

$$N^{(2)-}(\psi(0))=\{Q\in B_0(X):Q(x)+\beta^2\ge 0 \text{ for } (x,\beta)\in\gamma^-((0,0);uep\ \psi)\ .$$

Theorem 4. *Let X be a Hilbert space, f be a {2}- lipschitz function at the point x_0 with a constant K,* $g_1(x)=f(x_0+x)-2f(x_0)+f(x_0-x)+K\|x\|^2$, $g_2(x)=$ $=f(x_0+x)-2f(x_0)+f(x_0-x)-K\|x\|^2$. *Then*

$$\check{D}_2f(x_0)=\left(N^{(2)+}(g_1(0))-K\|x\|^2\right)\cap\left(N^{(2)-}(g_2(0))+K\|x\|^2\right).$$

Proof. If $Q\in\check{D}_2f(x_0)$, then $f^{(2)+}(x_0;x)\ge Q(x)$. It is clear, that $\overline{\lim_{\lambda\downarrow 0}}\frac{g_1(\lambda x)}{\lambda^2}=$ $=f^{(2)+}(x_0;x)+K\|x\|^2\ge 0$. Assume $\alpha=\sqrt{f^{(2)+}(x_0;x)+K\|x\|^2}$ we get, that there exist $\eta>0$ and $0(\lambda):[0,\eta]\to R_+$, where $\frac{0(\lambda)}{\lambda}\to 0$ for $\lambda\downarrow 0$, that $(\lambda x,\lambda\alpha+0(\lambda))\in uep\ g_1$.

Since $Q(x)+K\|x\|^2\le f^{(2)+}(x_0;x)+K\|x\|^2$, then we get $Q(x)+K\|x\|^2\in$ $\in N^{(2)+}(g_1(0))$.

It is clear that $\lim_{\lambda\downarrow 0}\frac{g_2(\lambda x)}{\lambda^2}=f^{(2)-}(x_0;x)-K\|x\|^2\le 0$. Assuming $\beta=\sqrt{K\|x\|^2-f^{(2)-}(x_0;x)}$ similar to lemma 13 we get that there exist $\eta>0$ and $0(\lambda):[0,\eta]\to R_+$, where $\frac{0(\lambda)}{\lambda}\to 0$ for $\lambda\downarrow 0$, that $(\lambda x,\lambda\beta+0(\lambda))\in lep\ g$ for $\lambda\in(0,\eta]$. Besides, $Q(x)-K\|x\|^2+K\|x\|^2-f^{(2)-}(x_0;x)\ge 0$, i.e. $Q(x)-K\|x\|^2\in N^{(2)-}(g_2(0))$. Thus we get

$$Q(x)\in\left(N^{(2)+}(g_1(0))-K\|x\|^2\right)\cap\left(N^{(2)-}(g_2(0))+K\|x\|^2\right).$$

Conversely, if $Q(x)\in\left(N^{(2)+}(g_1(0))-K\|x\|^2\right)\cap\left(N^{(2)-}(g_2(0))+K\|x\|^2\right)$, then $Q(x)+K\|x\|^2\in N^{(2)+}(g_1(0))$, $Q(x)-K\|x\|^2\in N^{(2)-}(g_2(0))$. Therefore $Q(x)+K\|x\|^2-\alpha^2\le 0$ for $(x,\alpha)\in\gamma^+((0,0);uep\ g_1)$. Since $\alpha^2\ge f^{(2)+}(x_0;x)+K\|x\|^2$, then $Q(x)+K\|x\|^2-f^{(2)+}(x_0;x)-K\|x\|^2\le 0$ or $f^{(2)+}(x_0;x)\ge Q(x)$. It is verified analogously that $f^{(2)-}(x_0;x)\le Q(x)$. Let $g_z(x)=f(z+x)-2f(z)+f(z-x)$, therefore $Q(x)\in\check{D}_2f(x_0)$. The theorem is proved.

$$\Gamma^+(g(0))=\left\{(x,\alpha)\in X\times R_+:\forall n \text{ exist } \exists\eta_n>0, \text{ that } \left(\lambda x,\lambda\alpha+\frac{\lambda}{\sqrt{n}}\right)\in uep\ g_z \right.$$

$$\left. \text{for } 0\le\lambda\le\eta_n,\ z\in B(x_0;\eta_n)\ \right\},$$

$$\Gamma^-(g(0))=\left\{(x,\beta)\in X\times R_+:\forall n \text{ exist } \exists\eta_n>0, \text{ that } \left(\lambda x, \lambda\beta+\frac{\lambda}{\sqrt{n}}\right)\in lep\, g_z \right.$$

$$\left. \text{for } 0\le\lambda\le\eta_n,\ z\in B(x_0;\eta_n)\ \right\},$$

$$N^{[2]+}(g(0))=\{Q\in B_0(X):Q(x)-\alpha^2\le 0 \text{ for } (x,\alpha)\in\Gamma^+(g(0))\},$$

$$N^{[2]-}(g(0))=\{Q\in B_0(X):Q(x)+\beta^2\ge 0 \text{ for } (x,\beta)\in\Gamma^-(g(0))\}.$$

Theorem 5. *If X is a Hilbert space, f is a {2}-lipschitz function in the vicinity of the point x_0 with a constant K, $g_{1z}(x)=f(z+x)-2f(z)+f(z-x)+K\|x\|^2$, $g_{2z}(x)=f(z+x)-2f(z)+f(z-x)-K\|x\|^2$, then*

$$D_2f(x_0)=\left(N^{[2]+}(g_1(0))-K\|x\|^2\right)\cap\left(N^{[2]-}(g_2(0))+K\|x\|^2\right).$$

Theorem 5 is proved analogously to theorem 4.

Consider the space $B_0(X)$ with a topology $\sigma(B_0(X),X)$. We shall say that Q_n converges weakly to Q, if $Q_n(x)$ converges to $Q(x)$ for any $x\in X$. It is known that if X is finite-dimensional, then we can identify $B(X^2,R)$ and $L(X,X)$.

Let $C\subset X$. The set $D_2^Cf(x)=\{Q\in B_0(X):Q_i\in D_2f(x_i), x_i\in C, x_i\to x, Q$ is a limit point the sequence $Q_i\}$ calls the C-relative bi-differential of the function f at the point x.

The following lemma immediately follows from the definition $D_2^Cf(x_0)$.

Lemma 12. *If X is finite-dimensional and f θ-bi-lipschitz function in the vicinity x_0, then*

1) *$D_2^Cf(x_0)$ is a closed subset of $D_2f(x_0)$,*
2) *$D_2^Cf(x_0)=D_2f(x_0)$, if $x_0\in\operatorname{int}C$; $D_2^Cf(x_0)=\varnothing$, if $(x_0+\varepsilon B)\cap C=\varnothing$ for some $\varepsilon>0$,*
3) *the mapping $D_2^Cf(x_0)$ is upper semicontinuous in x_0.*

Remark 1. If the function f satisfies the θ-two- lipschitz , weak θ- bi-lipschitz or weak θ-two-lipschitz condition in the vicinity x_0, then the statement Theorem 1,2, Corollary 1 and Lemma 7 is also valid.

Theorem 6. *If the point x_0 is a local minimum of the function f, then $0\in\check{D}_2^+f(x_0)$.*

It is clear, that $\check{D}_2^+f(x_0)\subset D_2^+f(x_0)\subset d_2^+f(x_0)$, where $d_2^+f(x_0)=\{Q\in B_0(X)$: $f^{[2]}(x_0;x,x)\ge Q(x), x\in X\}$.

Consider a subdifferential of higher order which is a direct generalization of the second order subdifferential.

Assume

$$f^{[n]+}(x_0;x)=\varlimsup_{\substack{z\to x_0\\ \lambda\downarrow 0}}\frac{\sum_{k=0}^{n}(-1)^kC_n^kf(z+(n-k)\lambda x)}{\lambda^n},$$

$$f^{|n|-}(x_0;x)=\varliminf_{\substack{z\to x_0\\ \lambda\downarrow 0}}\frac{\sum_{k=0}^{n}(-1)^k C_n^k f(z+(n-k)\lambda x)}{\lambda^n},$$

$$f^{(n)+}(x_0;x)=\varlimsup_{\lambda\downarrow 0}\frac{\sum_{k=0}^{n}(-1)^k C_n^k f\left(x_0+\left(\frac{n}{2}-k\right)\lambda x\right)}{\lambda^n},$$

$$f^{(n)-}(x_0;x)=\varliminf_{\lambda\downarrow 0}\frac{\sum_{k=0}^{n}(-1)^k C_n^k f\left(x_0+\left(\frac{n}{2}-k\right)\lambda x\right)}{\lambda^n}.$$

The symmetric functional $x_n^*\in N(X^n,R)$ satisfying the inequality

$$f^{[n]-}(x_0;x)\le x_n^*(x,\dots,x)\le f^{[n]+}(x_0;x),$$
$$\left(f^{(n)-}(x_0;x)\le x_n^*(x,\dots,x)\le f^{(n)+}(x_0;x)\right)$$

we call the $[n]$- gradient ((n)-gradient) of the function f at the point x_0, and the set of $[n]$-gradients ((n)-gradients) at the point x_0 we call the $[n]$- differential ((n)-differential) of the function f at the point x_0 and denote $D_n f(x_0)$ $\left(\check{D}_n f(x_0)\right)$.

Assume $C_n^f(z;x,\lambda)=\sum_{k=0}^{n}(-1)^k C_n^k f(z+(n-k)\lambda x)$,

$$S_n^f(z;x,\lambda)=\sum_{k=0}^{n}(-1)^k C_n^k f\left(x_0+\left(\frac{n}{2}-k\right)\lambda x\right).$$

The function f we call $[n,\theta]$- lipschitz ((n,θ)- lipschitz) with constant K in the vicinity x_0, if for some $\varepsilon>0$ it is fulfilled the following condition

$$\left|C_n^f(z;x,1)-C_n^f(z;y,1)\right|\le K\|x-y\|^{\theta}\left(\|x\|+\|y\|\right)^{n-\theta},$$
$$\left(\left|S_n^f(z;x,1)-S_n^f(z;y,1)\right|\le K\|x-y\|^{\theta}\left(\|x\|+\|y\|\right)^{n-\theta}\right) \quad (11)$$

for $x,y\in\varepsilon B$, $z\in x_0+\varepsilon B$, $0<\theta<n$.

If for $z=x_0$ the relation (11) is fulfilled, then the function f we call $[n,\theta]$-lipschitz ((n,θ)- lipschitz) with a constant K at the point x_0.

Theorem 7. *If f $[n,\theta]$- lipschitz function with a constant K in the vicinity u, then $f^{|n|+}(u,x)$ is upper semicontinuous as a function (u,x), and as a function only x satisfies the condition*

$$\left|f^{|n|+}(u;x)-f^{|n|+}(u;y)\right|\le K\|x-y\|^{\theta}\left(\|x\|+\|y\|\right)^{n-\theta}.$$

The corresponding statement is valid for $f^{|n|-}(u;x)$.

Theorem 8. *If X is finite-dimensional and f is a $[n,\theta]$- lipschitz function in the vicinity x_0, then $D_n f(u)$ is upper semicontinuous in x_0.*

Let $\psi:X\to R$. Assume

$$n-ep_{+}\psi=\{(x,\alpha)\in X\times R_+:\psi(x)\le\alpha^n\},$$
$$n-ep_{-}\psi=\{(x,\beta)\in X\times R_+:\psi(x)\ge-\beta^n\},$$

$$K_n^+(x_0,\psi(x_0))=\{(x,\alpha)\in X\times R_+:(x_0+\lambda x+0_1(\lambda),\psi(x_0)+\lambda\alpha+0_2(\lambda))\in n-ep_{\,+}\psi \text{ for }$$
$$\lambda\in(0,\eta_{x,\alpha}),\eta_{x,\alpha}>0,\frac{0_1(\lambda)}{\lambda^n}\to 0,\frac{0_2(\lambda)}{\lambda}\to 0 \text{ for } \lambda\downarrow 0\Big\},$$
$$K_n^-(x_0,\psi(x_0))=\{(x,\beta)\in X\times R_+:(x_0+\lambda x+0_1(\lambda),\psi(x_0)+\lambda\beta+0_2(\lambda))\in n-ep_{\,-}\psi \text{ for }$$
$$\lambda\in(0,\eta_{x,\beta}),\eta_{x,\beta}>0,\frac{0_1(\lambda)}{\lambda^n}\to 0,\frac{0_2(\lambda)}{\lambda}\to 0 \text{ for } \lambda\downarrow 0\Big\},$$
$$d_n(x,\alpha)=\inf\{\|x-y\|+|\alpha-\beta|^n:(y,\beta)\in n-ep_{\,+}\psi\}.$$

Lemma 13. $(x,\alpha)\in K_n^+(x_0,\psi(x_0))$ *if and only if*

$$\overline{\lim_{\lambda\downarrow 0}}\frac{d_n((x_0,\psi(x_0))+\lambda(x,\alpha))}{\lambda^n}=0.$$

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