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SOME QUESTIONS OF RIESZ SUMMABILITY OF MULTIPLE FOURIER INTEGRALS

Abstract

In this paper we study even Riesz summability of multiple Fourier integrals in Euclidean space. We have proved the following theorem.

Theorem. *Let $f(x)\in L_1(R^k)$ have packaged support space R^k. If $0<\sigma<1$,*

$$|f(x+h)-f(x)|\le C\frac{|h|^\alpha}{\min\left[r^\sigma(x),r^\sigma(x+h)\right]}\qquad(\alpha>0),$$

where $r(x)=\inf_{y\in\Phi}|x-y|$, $x,y,h\in R^k$, *then by* $\delta=p+(k-1)/2+\eta$ $(0<\eta<\alpha/2)$ *complete equation*

$$\widetilde{S}_R^\delta(x,f)-2^{\frac{k-2}{2}}\Gamma\left(\frac{k}{2}\right)f(x)=O\left(\frac{1}{R^{\alpha-\eta}}\right),\ R\to\infty$$

evenly relatively $x\in G$, *on every packaged* $G\subset R^k$, *where* $G\cap(\Phi\cup N(\Phi))=\varnothing$.

Let $f(x)=f(x_1,x_2,\dots,x_k)\in L_1(R^k)$ be a periodic function with period 2π in each variable.

Fourier series of the function $f(x)$ is written as

$$f(x)\sim\sum a_{n_1\dots n_k}e^{i(n_1x_1+\dots+n_kx_k)},\tag{1}$$

where

$$a_{n_1\dots n_k}=\frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}f(x)e^{-i(n_1x_1+\dots+n_kx_k)}$$

is a Fourier coefficient of this function.

For $\delta\ge0$ it is determined the sum

$$S_k^\delta(x,f)=\sum_{\nu\le R^2}\left(1-\frac{\nu^2}{R^2}\right)^\delta a_{n_1\dots n_k}e^{i(n_1x_1+\dots+n_kx_k)}\tag{2}$$

being the Riesz mean of order δ series (1), where $\nu^2=n_1^2+\dots+n_k^2$. If there exists a finite limit $\lim_{R\to\infty}S_R^\delta(x,f)$ then it is said that series (1) is summed up by a Riesz method at the point $x\in R^k$.

Bohner [1] has proved the following representation for the Riesz mean (2) of Fourier series of the function $f(x)$

$$S_R^\delta(x,f)=2^\delta\Gamma(\delta+1)R^k\int_0^\infty t^{k-1}f_x(t)V_{\delta+k/2}(tR)dt,\tag{3}$$

where $V_\nu(x)=\frac{J_\nu(x)}{x^\nu}$, $J_\nu(x)$ is a Bessel function and

$$f_x(t)=\frac{\Gamma(k/2)}{2(\pi)^{k/2}}\int_\Sigma f(x_1+t\xi_1,\dots,x_k+t\xi_k)d\Sigma_\xi$$

is a spherical mean of the mean function $f(x)$ in a sphere with a center at the point $x \in R^k$ and radius t, $\Sigma: \xi_1^2 + \cdots + \xi_k^2 = 1$ is a unique sphere.

K.Chandrasekharan [2] has generalized formula (3) as follows:

$$S_R^\delta(x,f) = \frac{2^{\delta-p}\Gamma(\delta+1)\Gamma(k/2)}{\Gamma(p+k/2)} R^{k+2p} \int_0^\infty t^{k+2p-1} f_p(x,t) V_{\delta+k/2+p}(tR)dt, \qquad (4)$$

where $\delta > h + \frac{k-1}{2}$, $p > 0$, h is the greatest member less than p

$$f_p(x,t) = \frac{2}{B(p;k/2)t^{2p+k-2}} \int_0^t \left(t^2 - s^2\right)^{p-1} s^{k-1} f(x,s)\, ds$$

is a spherical mean of order p of the function $f(x) \in L_1\left(R^k\right)$, $B(\alpha,\beta)$ is a Beta function.

In view of ([2], p.217)

$$R^{k+2p} \int_0^\infty t^{k+2p-1} V_{\delta+p+k/2}(t\, R)d\,t = \frac{\Gamma(p+k/2)}{2^{\delta-p-k/2+1}\Gamma(\delta+1)}$$

we find the representation

$$S_R^\delta(x,f) - 2^{(k-2)/2}\Gamma(k/2)f(x) =$$

$$= \frac{2^{\delta-p}\Gamma(\delta+1)\Gamma(k/2)}{\Gamma(p+k/2)} R^{k+2p} \int_0^\infty t^{k+2p-1}\left[f_p(x,t) - f(x)\right] V_{\delta+k/2+p}(tR)d\,t.$$

In paper [2] Chandrasekhan has studied the Riesz summability of multiple Fourier series.

In particular, Chandrasekhan has proved that if

$$f_p(x,t) - f(x) = O\left(t^\alpha\right), \quad t \to 0, \quad \alpha > 0,$$

then for $\delta = p + \frac{k-1}{2} + \beta$ $(\beta > 0)$

$$\lim_{R\to\infty} S_R^\delta(x,f) = 2^{(k-2)/2}\Gamma(k/2)f(x)$$

uniformly with respect to $x \in R^k$.

In the present paper this problem is studied for multiple Fourier integrals in a more general form.

Let $f(x) \in L_1\left(R^k\right)$ and

$$f(x) \sim \int_{R^k} \hat{f}(u) e^{i(x,u)} du \qquad (5)$$

be its expansion in Fourier integral, where

$$\hat{f}(u) = \frac{1}{(2\pi)^k} \int_{R^k} f(x) e^{-i(x,u)} dx, \ x \in R^k, \ u \in R^k$$

is its Fourier transformation.

$$\tilde{S}_R^\delta(x,f) = \int_{|u|<R} \left(1 - \frac{|u|^2}{R^2}\right)^\delta \hat{f}(u) e^{i(x,u)} du \qquad (6)$$

is the Riesz's spherical mean of order $\delta \geq 0$ for Fourier integral (5).

Note that representations (3) and (4) are valid with respect to the Riesz spherical mean for Fourier's integral.

The following problem is set.

Let the conditions of K. Chandrasekhan hold for the function $f(x)\in L_1(R^k)$ everywhere in R^k, except the points of some twice differentiable hypersurface $\Phi\subset R^k$. Then what can we say about the domain of uniform Riesz summability of Fouriers's multiple integrals in the space R^k.

Denote by $N(\Phi)$ a set of curvature centers of rounding points of the hypersurface Φ. It holds

Theorem. *Let $f(x)\in L_1(R^k)$ have a compact support in R^k. If for $0<\sigma<1$ it is fulfilled the inequality*

$$|f(x+h)-f(x)|\le C\frac{|h|^\alpha}{\min\left[r^\sigma(x),r^\sigma(x+h)\right]}\ (\alpha>0),$$

where $r(x)=\inf\limits_{y\in\Phi}|x-y|$, $x, y, h\in R^k$, then for $\delta=p+\frac{k-1}{2}+\eta$ $\left(0<\eta<\frac{\alpha}{2}\right)$ it holds the equality

$$\tilde{S}_R^\delta(x,f)-2^{\frac{k-2}{2}}\Gamma\left(\frac{k}{2}\right)f(x)=O\left(\frac{1}{R^{\alpha-\eta}}\right),\quad R\to\infty$$

uniform with respect to $x\in G$, on any compact $G\subset R^k$, having no generic points with the set $\Phi\cup N(\Phi)$.

We use the method developed in papers [3] and [5] to prove the theorem.

We prove the following lemmas beforehand.

Let $\Phi\subset R^k$ be a twice continuously-differentiable compact hypersurface. $\vec{r}(u)=\vec{r}(u_1,u_2,..,u_{k-1})$ is a radius-vector of hypersurface.

Lemma 1. *Let $A\in\Phi$ be a point of a hypersurface. If $dF(A)=0$ and $d^2F(A)=0$, where $F(u_1,u_2,...,u_{k-1})=|\vec{r}(u_1,u_2,...,u_{k-1})|^2$, then the point $A\in\Phi$ is a rounding point of the hypersurface Φ.*

Proof. We have

$$dF(A)=\sum_{j=1}^{k-1}\frac{\partial F(A)}{\partial u_j}du_j=2\sum_{j=1}^{k-1}\left(\frac{\partial\vec{r}}{\partial u_j},\vec{r}\right)\partial u_j, \tag{7}$$

$$d^2F(A)=2\sum_{j=1}^{k-1}\sum_{i=1}^{k-1}\left(\frac{\partial^2\vec{r}}{\partial u_j\partial u_i},\vec{r}\right)du_jdu_i+2\sum_{j=1}^{k-1}\sum_{i=1}^{k-1}\left(\frac{\partial\vec{r}}{\partial u_j},\frac{\partial\vec{r}}{\partial u_i}\right)du_jdu_i. \tag{8}$$

Assume that $dF(A)=0$ and $d^2F(A)=0$ simultaneously. Then it follows from (7), that $\frac{\partial\vec{r}}{\partial u_j}\perp\vec{r}$, i.e. $\vec{r}\perp Q$, where Q is a tangent plane to the hypersurface Φ at the point $A\in\Phi$. As $\vec{m}\perp Q$, where $\vec{m}$ is a normal to the hypersurface at the point A, hence it follows that $\vec{r}=\lambda\vec{m}$. Taking this into account we find from (8):

$$\lambda\sum_{j=1}^{k-1}\sum_{i=1}^{k-1}\left(\frac{\partial^2\vec{r}}{\partial u_j\partial u_i},\vec{m}\right)du_jdu_i+\sum_{j=1}^{k-1}\sum_{i=1}^{k-1}\left(\frac{\partial\vec{r}}{\partial u_j},\frac{\partial\vec{r}}{\partial u_i}\right)du_jdu_i=0. \tag{9}$$

In view of

$$B(u,du)=\sum_{j=1}^{k-1}\sum_{i=1}^{k-1}\left(\frac{\partial^2\vec{r}}{\partial u_j\partial u_i},\vec{m}\right)du_jdu_i,\qquad G(u,du)=\sum_{j=1}^{k-1}\sum_{i=1}^{k-1}\left(\frac{\partial\vec{r}}{\partial u_j},\frac{\partial\vec{r}}{\partial u_i}\right)du_jdu_i$$

are the second and first quadratic forms respectively, we find from (9) that

$$K_N = \frac{B(u,du)}{G(u,du)} = -\frac{1}{\lambda} \tag{10}$$

i.e. at the point A on the hypersurface in all directions a normal curvature has the same value. This means that the point $A \in \Phi$ is a rounding point of the hypersurface $\Phi \subset R^k$. The lemma is proved.

Lemma 2. *Let* $A \in (\Phi \cap C_\rho)$, *where* C_ρ *is a sphere with a radius* ρ *and with a center at the point* $x_0 \in R^k$ *and let* $dF(A) = 0$ *and* $d^2F(A) = 0$. *In this case the curvature centers of the hypersurface* Φ *and the sphere* C_ρ *at the point* A *coincide.*

Proof. It follows from lemma 1 that the point A is a rounding point of the hypersurface Φ. At this point the equality (10) holds. Consequently, $\lambda = -R_N$, where R_N is a normal curvature radius of the hypersurface at the point A. Hence it follows

$$\vec{r}(u) = -R_N \vec{m} \quad \text{and} \quad |\vec{r}(u)| = R_N .$$

In other words, $\rho = R_N$. This means that the curvature and curvature radius of the hypersurface Φ and the sphere C_ρ at the point A coincide.

Denote by $\vec{r}(s)$ a radius-vector of arbitrary curve arranged on the hypersurface Φ and passing through the considered points, where s is a natural parameter of this curve. It is obvious that $F(s) = |\vec{r}(s)|^2$ is a twice-differentiable function. Assume that the function $F(s)$ satisfies the conditions of lemma 2-5 of paper [3] by E.M. Nikishin and G.I. Osmanov. By virtue of these lemmas there exists only the finite number of points $s_1, s_2, \ldots, s_n$ such that $F'(S) = 0$. Otherwise, the point x_0 will be the curvature point center of rounding points of the hypersurface Φ. But $x_0 \notin N(\Phi)$. The lemma is proved.

Suppose $\rho_j = |\vec{r}(s_j)|$ and $\gamma(\rho) = \min\limits_j |\rho - \rho_j|$.

Denote by $\Lambda(\rho,\omega)$ the distance from the point of the sphere C_ρ with a radius ρ and center at the point $x_0 \notin N(\Phi)$ up to the hypersurface Φ.

Lemma 3. *For* $0 < \delta < \rho/3$ *it holds the inequality*

$$mes\{\omega \in C_\rho : \Lambda(\rho,\omega) \le \delta\} \le C\delta\,\rho^{-1} + C\delta\rho^{3/2-k}\left[\gamma^{-1/2}(\rho-\delta) + \gamma^{-1/2}(\rho+\delta)\right], \tag{11}$$

where C *doesn't depend on* ρ *and* δ.

Proof. The curve lying in hypersurface and passing through the considered point of the hypersurface divide into finite number of curves p. Assume that $\Lambda_p(\rho,\omega)$ is the distance from the point (ρ,ω) to the curve l_p. In this case it is obvious that

$$\Lambda(\rho,\omega) = \min_p \Lambda_p(\rho,\omega).$$

Therefore

$$mes\{\omega : \Lambda(\rho,\omega) \le \delta\} \le \sum_p mes\{\omega : \Lambda_p(\rho,\omega) \le \delta\} . \tag{12}$$

Now estimate $\rho^{k-1} mes\{\omega : \Lambda_p(\rho,\omega) \le \delta\}$.

Draw the spheres $C_{\rho-\delta}$ and $C_{\rho+\delta}$. If the hypersurface has no point between the spheres $C_{\rho-\delta}$ and $C_{\rho+\delta}$, then

$$mes\{\omega : \Lambda_p(\rho,\omega) \le \delta\} = 0 .$$

Otherwise, a set of spherical points (ρ,ω), distant from the curve l_p not exceeding δ, represents a part of the sphere C_p. The measure of this part of the sphere doesn't exceed the measure of the part of hypersurface hit interior the ring-domain, plus $c\delta\rho^{k-2}$. Thus, we have

$$\rho^{k-1} mes\{\omega : \Lambda_p(\rho,\omega) \le \delta\} \le mes\{u \in \Phi : (\rho-\delta)^2 \le F(u) \le (\rho+\delta)^2\} + C\delta\,\rho^{k-2}.$$

On the hand we can write

$$mes\{u \in \Phi : (\rho-\delta)^2 \le F(u) \le (\rho+\delta)^2\} \le$$
$$\sup mes\{S : (\rho-\delta)^2 \le |\bar{r}(s)|^2 \le (\rho+\delta)^2\},$$

where supremum is taken along all curves lying on the hypersurface and passing through the considered points.

By virtue of lemma 6 of paper [3] it follows that

$$mes\{u \in \Phi : (\rho-\delta)^2 \le F(u) \le (\rho+\delta)^2\} \le C\delta\rho\{[\rho\gamma(\rho-\delta)]^{-1/2} + [\rho\gamma(\rho+\delta)]^{-1/2}\}.$$

Consequently,

$$\rho^{k-1} mes\{\omega : \Lambda_p(\rho,\omega) \le \delta\} \le C\delta\rho\{[\rho\gamma(\rho-\delta)]^{-1/2} + [\rho\gamma(\rho+\delta)]^{-1/2}\} + C\delta\rho^{k-2}.$$

Hence we find that

$$mes\{\omega : \Lambda_p(\rho,\omega) \le \delta\} \le C\delta\rho^{-1} + C\delta\rho^{k-2}\{[\rho\gamma(\rho-\delta)]^{-1/2} + [\rho\gamma(\rho+\delta)]^{-1/2}\}. \quad (13)$$

The validity of relation (11) follows from (12) and (13).

Lemma 4. *Let* $0<\sigma<1$, $0<a<\infty$, $\mu = \dfrac{2}{2\delta - 2p - k + 1}$. *Then the integral*

$$\int_0^a \left(\int_\Sigma \frac{d\omega}{\Lambda^\sigma(x,\rho,\omega)} \right)^\mu d\rho$$

uniformly converges with regard to x *on any compact* $G \subset R^k$, *having no generic points with the set* $\Phi \cup N(\Phi)$, *where* $\Lambda(x,\rho,\omega)$ *is the distance from the points of the sphere with the radius* ρ *and center at the point* $x \in G$ *to the hypersurface* Φ.

Since the compactum G has no generic points with the hypersurface, for any $x \in G$ at small values ρ, we have

$$\Lambda(x,\rho,\omega) \ge d, \quad \text{where} \quad d \in R_+.$$

Consequently, for the proof of lemma 4 it is sufficient to prove the uniform convergence of the integral

$$\int_d^a \left(\int_\Sigma \frac{d\omega}{\Lambda^\sigma(x,\rho,\omega)} \right)^\mu d\rho \quad \text{with respect to} \quad x \in G.$$

Choose the number $\beta > \dfrac{\mu}{2} > 1$, since $\sigma\beta < 1$. First estimate the inner integral

$$\int_\Sigma \frac{d\omega}{\Lambda^\sigma(x,\rho,\omega)}.$$

Put $E_n = \left\{\omega : \dfrac{\rho}{3^{n+1}} \le \Lambda(x,\rho,\omega) \le \dfrac{\rho}{3^n}\right\}$, $n = 1,2,3,\ldots,$

$$E_0 = \left\{\omega : \Lambda(x,\rho,\omega) \ge \frac{\rho}{3}\right\}.$$

Then, by virtue of lemma 3 we have

$$mE_n \le C\frac{\rho}{3^n}\rho^{-1} + C\frac{\rho}{3^n}\rho^{3/2-k}\left\{\gamma^{-1/2}\left[\rho\left(1-\frac{1}{3^n}\right)\right]+\gamma^{-1/2}\left[\rho\left(1+\frac{1}{3^n}\right)\right]\right\}, \qquad (14)$$

$$mE_0 \le \frac{2\pi^{k/2}}{\Gamma(k/2)}. \qquad (15)$$

By virtue of Hölder inequality we have:

$$\int_\Sigma \frac{d\omega}{\Lambda^{\sigma}(x,\rho,\omega)} \le C\left(\int_\Sigma \frac{d\omega}{\Lambda^{\sigma\beta}(x,\rho,\omega)}\right)^{1/\beta}. \qquad (16)$$

Considering the relations (14) and (15) we find

$$\int_\Sigma \frac{d\omega}{\Lambda^{\sigma\beta}(x,\rho,\omega)} = \sum_{n=1}^{\infty}\int_{E_n}\frac{d\omega}{\Lambda^{\sigma\beta}(x,\rho,\omega)} + \int_{E_0}\frac{d\omega}{\Lambda^{\sigma\beta}(x,\rho,\omega)} \le \frac{C}{\rho^{\sigma\beta}} +$$

$$+C\sum_{n=1}^{\infty}\frac{3^{n\sigma\beta}}{\rho^{\sigma\beta}}\left\{\frac{C}{3^n}+\frac{C}{3^n}\rho^{5/2-k}\left(\gamma^{-1/2}\left[\rho\left(1-\frac{1}{3^n}\right)\right]+\gamma^{-1/2}\left[\rho\left(1+\frac{1}{3^n}\right)\right]\right)\right\} =$$

$$= \frac{C}{\rho^{\sigma\beta}} + C\sum_{n=1}^{\infty}\left\{\frac{\rho^{-\sigma\beta}}{3^{n(1-\sigma\beta)}} + \frac{1}{3^{n(1-\sigma\beta)}}\rho^{5/2-k-\sigma\beta}\left(\gamma^{-1/2}\left[\rho\left(1-\frac{1}{3^n}\right)\right]+\gamma^{-1/2}\left[\rho\left(1+\frac{1}{3^n}\right)\right]\right)\right\} =$$

$$= \frac{C}{\rho^{\sigma\beta}} + C\sum_{n=1}^{\infty}\frac{\rho^{-\sigma\beta}}{3^{n(1-\sigma\beta)}} + C\sum_{n=1}^{\infty}\frac{\rho^{5/2-k-\sigma\beta}}{3^{n(1-\sigma\beta)}}\left\{\gamma^{-1/2}\left[\rho\left(1-\frac{1}{3^n}\right)\right]+\gamma^{-1/2}\left[\rho\left(1+\frac{1}{3^n}\right)\right]\right\}.$$

Since $\sigma\beta < 1$, then the series $\sum_{n=1}^{\infty}\frac{1}{3^{n(1-\sigma\beta)}}$ converges. Taking this into account we find that

$$\int_\Sigma \frac{d\omega}{\Lambda^{\sigma\beta}(x,\rho,\omega)} \le \frac{C}{\rho^{\sigma\beta}} + C\sum_{n=1}^{\infty}\frac{\rho^{5/2-k-\sigma\beta}}{3^{n(1-\sigma\beta)}}\left\{\gamma^{-1/2}\left[\rho\left(1-\frac{1}{3^n}\right)\right]+\gamma^{-1/2}\left[\rho\left(1+\frac{1}{3^n}\right)\right]\right\}. \qquad (17)$$

By applying Minkowskii's inequality and considering (16) and (17) we find

$$\left\{\int_d^a\left(\int_\Sigma \frac{d\omega}{\Lambda^{\sigma}(x,\rho,\omega)}\right)^{\mu} d\rho\right\}^{\beta/\mu} \le \left\{\int_d^a\left(\int_\Sigma \frac{d\omega}{\Lambda^{\sigma\beta}(x,\rho,\omega)}\right)^{\mu/\beta} d\rho\right\}^{\beta/\mu} \le$$

$$\le \left\{\int_d^a\left(\frac{C}{\rho^{\sigma\beta}} + C\sum_{n=1}^{\infty}\frac{\rho^{5/2-k-\sigma\beta}}{3^{n(1-\sigma\beta)}}\left\{\gamma^{-1/2}\left[\rho\left(1-\frac{1}{3^n}\right)\right]+\gamma^{-1/2}\left[\rho\left(1+\frac{1}{3^n}\right)\right]\right\}\right)^{\mu/\beta} d\rho\right\}^{\beta/\mu} \le$$

$$\le \left\{\int_d^a\frac{d\rho}{\rho^{\mu\sigma}}\right\}^{\beta/\mu} + \sum_{n=1}^{\infty}\frac{1}{3^{n(1-\sigma\beta)}}\left\{\left[\int_d^a\frac{\rho^{(5/2-k-\sigma\beta)\mu/\beta}}{\gamma^{\mu/2\beta}\left[\rho\left(1-1/3^n\right)\right]}d\rho\right]^{\beta/\mu} + \left[\int_d^a\frac{\rho^{(5/2-k-\sigma\beta)\mu/\beta}}{\gamma^{\mu/2\beta}\left[\rho\left(1+1/3^n\right)\right]}d\rho\right]^{\beta/\mu}\right\}.$$

Making a change of variable by the formula $\rho\left(1-\frac{1}{3^n}\right)=t$ in the first integral and $\rho\left(1+\frac{1}{3^n}\right)=t$ in the second integral we find:

$$\left\{\int_d^a\left(\int_\Sigma \frac{d\omega}{\Lambda^{\sigma}(x,\rho,\omega)}\right)^{\mu} d\rho\right\}^{\beta/\mu} \le \left\{\int_d^a\frac{d\rho}{\rho^{\mu\sigma}}\right\}^{\beta/\mu} + \sum_{n=1}^{\infty}\frac{1}{3^{n(1-\sigma\beta)}}\left\{\int_{2d/3}^{2a}\frac{dt}{t^{\sigma\mu-(5/2-k)\mu/\beta}\left[\gamma(t)\right]^{\mu/2\beta}}\right\}^{\beta/\mu}.$$

For the function $\gamma(t)$ we can write the following inequality

$$\frac{1}{[\gamma(t)]^{\mu/2\beta}} \le \sum_p \frac{1}{\left|t-\rho_p\right|^{\mu/2\beta}} = t^{-\mu/2\beta} + \sum_{p:\rho_p \ne 0} \frac{1}{\left|t-\rho_p\right|^{\mu/2\beta}}.$$

Since $\frac{\mu}{2\beta} < 1$, then $\int\limits_{2d/3}^{2a} \frac{dt}{\left|t-\rho_p\right|^{\mu/2\beta}} \le \infty$.

Consequently, $\int\limits_{2d/3}^{2a} \frac{dt}{t^{\sigma\mu+(k-5/2)\mu/\beta}[\gamma(t)]^{\mu/2\beta}} < +\infty$. Then we conclude that $\int\limits_d^a \left(\int\limits_\Sigma \frac{d\omega}{\Lambda^\sigma(x,\rho,\omega)}\right)^\mu d\rho < +\infty$ uniformly with respect to $x \in G$, where $G \cap (\Phi \cup N(\Phi)) = \varnothing$. The lemma 4 is proved.

Put $\varphi_x(t) = \frac{f_p(x,t) - f(x)}{t^{\delta-p-k/2+3/2}}$, $\delta = p + \frac{k-1}{2} + \eta$.

Lemma 5. *Let the conditions of the theorem be fulfilled. Then for* $h \to 0$ $\int\limits_0^\infty \left|\varphi_x(t+h) - \varphi_x(t)\right| dt = O\left(h^{\alpha-2\eta}\right)$ $(\eta < \alpha/2)$ *is uniform with respect to* $x \in G$.

Proof. Since $\frac{2}{B(p,k/2)t^{2p+k-2}} \int\limits_0^t \left(t^2 - s^2\right)^{p-1} s^{k-1} ds = 1$ we have that

$$f_p(x,t) - f(x) = \frac{2}{B(p,k/2)t^{2p+k-2}} \int\limits_0^t \left(t^2 - s^2\right)^{p-1} s^{k-1} [f(x,s) - f(x)] ds.$$

It follows from the conditions of the theorem that

$$\left|f(x,s) - f(x)\right| \le Cs^\alpha \int\limits_\Sigma \frac{d\omega}{\Lambda^\sigma(x,s,\omega)}.$$

In view of $\Lambda(x,s,\omega) \ge \Lambda(x,t,\omega)$ for $s \le t$ hence we find:

$$\left|f_p(x,t) - f(x)\right| \le \frac{C}{B(p,k/2)t^{2p+k-2}} \int\limits_0^t \left(t^2 - s^2\right)^{p-1} s^{k-1} \left(s^\alpha \int\limits_\Sigma \frac{d\omega}{\Lambda^\sigma(x,s,\omega)}\right) ds \le$$

$$\le Ct^\alpha \int\limits_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t,\omega)} \cdot \frac{2t^{-2p-k+2}}{B(p,k/2)} \int\limits_0^t \left(t^2 - s^2\right)^{p-1} s^{k-1} ds = Ct^\alpha \int\limits_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t,\omega)}.$$

We have

$$\int\limits_0^\infty \left|\varphi_x(t+h) - \varphi_x(t)\right| dt = \int\limits_0^h \left|\varphi_x(t+h) - \varphi_x(t)\right| dt + \int\limits_h^\infty \left|\varphi_x(t+h) - \varphi_x(t)\right| dt = A_1 + A_2. \quad (18)$$

Estimate the integrals A_1 and A_2 separately

$$A_1 = \int\limits_0^h \left|\varphi_x(t+h) - \varphi_x(t)\right| dt \le \int\limits_0^h \frac{\left|f_p(x,t+h) - f(x)\right|}{(t+h)^{\delta-p-k/2+3/2}} dt + \int\limits_0^h \frac{\left|f_p(x,t) - f(x)\right|}{t^{\delta-p-k/2+3/2}} dt = B_1 + B_2.$$

Taking into account $\left|f_p(x,t+h) - f(x)\right| \le C(t+h)^\alpha \int\limits_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t+h,\omega)}$ we find for the integral B_1

$$B_1 \le C\int\limits_0^h (t+h)^{\alpha-\delta+p+(k-3)/2} \left(\int\limits_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t+h,\omega)}\right) dt.$$

By applying Hölder's inequality for $\frac{1}{\lambda}+\frac{1}{\mu}=1$, where $\mu=\frac{2}{2\delta-2p-k+1}$ and $\lambda=\frac{2}{1-2\delta+2p+k}$ we find

$$B_1 \le C\left\{\int_0^h (t+h)^{\frac{2\alpha-2\delta+2p+k-3}{2}\lambda}dt\right\}^{1/\lambda}\cdot\left\{\int_0^h\left(\int_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t+h,\omega)}\right)^\mu dt\right\}^{1/\mu} \le$$

$$\le C\left\{\int_0^h (t+h)^{\frac{2\alpha-2\delta+2p+k-3}{2}\lambda}dt\right\}^{1/\lambda} = O\left(h^{\alpha-2\eta}\right). \qquad (19)$$

By the same way we prove

$$B_2=\int_0^h \frac{\left|f_p(x,t)-f(x)\right|}{t^{\delta-p-k/2+3/2}}dt = O\left(h^{\alpha-2\eta}\right). \qquad (20)$$

Considering (19) and (20) we find $A_1=O\left(h^{\alpha-2\eta}\right)$ for $h\to 0$ uniformly with respect to $x\in G$, where $G\cap(\Phi\cup N(\Phi))=\varnothing$.

Transform the integral A_2 by the following way:

$$A_2=\int_h^\infty\left|\frac{f_p(x,t+h)-f(x)}{(t+h)^{\delta-p-(k-3)/2}}-\frac{f_p(x,t)-f(x)}{t^{\delta-p-(k-3)/2}}\right|dt \le \int_h^\infty \frac{\left|f_p(x,t+h)-f_p(x,t)\right|}{(t+h)^{\delta-p-(k-3)/2}}dt+$$

$$\int_h^\infty\left|f_p(x,t)-f(x)\right|\left|\frac{1}{(t+h)^{\delta-p-(k-3)/2}}-\frac{1}{t^{\delta-p-(k-3)/2}}\right|dt=E_1+E_2.$$

By virtue of the conditions of the theorem we have

$$E_1\le Ch^\alpha\int_h^{\rho_0}(t+h)^{-\delta+p+(k-3)/2}\left(\int_\Sigma\frac{d\omega}{\Lambda^\sigma(x,t,\omega)}\right)dt.$$

By applying Hölder's inequality and considering lemma 4, we find

$$E_1\le Ch^\alpha\left(\int_h^{\rho_0}(t+h)^{(-\delta+p+(k-3)/2)\lambda}dt\right)^{1/\lambda}\left(\int_h^{\rho_0}\left(\int_\Sigma\frac{d\omega}{\Lambda^\sigma(x,t,\omega)}\right)^\mu dt\right)^{1/\mu}\le$$

$$\le Ch^\alpha\left(\int_h^{\rho_0}(t+h)^{(-\delta+p+(k-3)/2)\lambda}dt\right)^{1/\lambda}=O\left(h^{\alpha-2\eta}\right)$$

uniformly with respect to $x\in G$.

Now it remains to estimate the integral E_2.

By viewing that $f(x)$ has a compact support in R_k, represent the integral E_2 as

$$E_2=\int_h^{\rho_0}\left|f_p(x,t)-f(x)\right|\left|\frac{1}{(t+h)^{\delta-p-k/2+3/2}}-\frac{1}{t^{\delta-p-k/2+3/2}}\right|dt+$$

$$+\int_{\rho_0}^\infty\left|f(x,t)\right|\left|\frac{1}{(t+h)^{\delta-p-k/2+3/2}}-\frac{1}{t^{\delta-p-k/2+3/2}}\right|dt=F_1+F_2.$$

It is easy to show that

$$\left|\frac{1}{(t+h)^{\delta-p-k/2+3/2}}-\frac{1}{t^{\delta-p-k/2+3/2}}\right|\le\frac{Ch}{t^{\delta+(5-k-2p)/2}}.$$

Hence it follows that

$$F_1 \le Ch\int_h^{\rho_0}\frac{|f_p(x,t)-f(x)|}{t^{\delta+(5-k-2p)/2}}dt \le Ch\int_h^{\rho_0} t^{\alpha-\delta-(5-k-2p)/2}\left(\int_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t,\omega)}\right)dt \le$$

$$\le Ch\left(\int_h^{\rho_0} t^{(\alpha-\delta-(5-k-2p)/2)\lambda}dt\right)^{1/\lambda}\left(\int_h^{\rho_0}\left(\int_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t,\omega)}\right)^\mu dt\right)^{1/\mu} \le$$

$$\le Ch\left(\int_h^{\rho_0} t^{(\alpha-\delta-(5-k-2p)/2)\lambda}dt\right)^{1/\lambda} = Ch\cdot h^{\alpha-2\eta-1} = Ch^{\alpha-2\eta}.$$

uniformly with respect to $x \in G$.

We have for the integral F_2:

$$F_2 \le C\int_{\rho_0}^{\infty}\frac{h}{t^{\delta+(5-k-2p)/2}}dt = Ch.$$

Thus, we proved that

$$A_2 = O\left(h^{\alpha-2\eta}\right) \quad \text{for} \quad h \to 0$$

uniformly with respect to $x \in G$.

Then from (18) follows, that

$$\int_0^\infty |\varphi_x(t+h)-\varphi_x(t)|\,dt = O\left(h^{\alpha-2\eta}\right) \text{ for } \quad h \to 0$$

uniformly with respect to $x \in G$. The lemma is proved.

Now we proved the theorem. We have:

$$\widetilde{S}_R^\delta(x,f) - 2^{(k-2)/2}\Gamma(k/2)f(x) =$$

$$= CR^{k+2p}\int_0^\infty t^{k+2p-1}\left[f_p(x,t)-f(x)\right]V_{\delta+p+k/2}(t\,R)dt =$$

$$= CR^{k+2p}\int_0^{1/R} t^{k+2p-1}\left[f_p(x,t)-f(x)\right]V_{\delta+p+k/2}(t\,R)dt +$$

$$+ CR^{k+2p}\int_{1/R}^\infty t^{k+2p-1}\left[f_p(x,t)-f(x)\right]V_{\delta+p+k/2}(t\,R)dt = M_1 + M_2.$$

First, we estimate the integral M_1. For $0 \le t \le \frac{1}{R}$ we have

$$\left|V_{\delta+p+k/2}\right| = \frac{\left|J_{\delta+p+k/2}(tR)\right|}{(tR)^{\delta+p+k/2}} = \frac{O\left((tR)^{\delta+p+k/2}\right)}{(tR)^{\delta+p+k/2}} = O(1).$$

Considering this inequality and applying Hölder's inequality for $\frac{1}{\lambda}+\frac{1}{\mu}=1$ we find

$$M_1 \le CR^{k+2p}\int_0^{1/R} t^{\alpha+k+2p-1}\left(\int_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t,\omega)}\right)dt \le$$

$$\le CR^{k+2p}\left(\int_0^{1/R} t^{(\alpha+k+2p-1)\lambda}dt\right)^{1/\lambda}\left(\int_0^{1/R}\left(\int_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t,\omega)}\right)^\mu dt\right)^{1/\mu} \le$$

$$\le CR^{k+2p}\left(\int_0^{1/R} t^{(\alpha+k+2p-1)\lambda}dt\right)^{1/\lambda} = CR^{k+2p}\cdot R^{-(\alpha+k+2p-1)-1/\lambda} = O\left(\frac{1}{R^{\alpha-\eta}}\right)\ R\to\infty .$$

uniformly with respect to $x\in G$.

To integrate the integral M_2 we use the the asymptotic representation of Bessel's function

$$J_\nu(x)=\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)+O\left(x^{-3/2}\right)\ \text{ for }\ x\to\infty .$$

Viewing this representation, expand the integral M_2 in sum of two integrals as follows

$$M_2\le CR^{k+2p}\int_{1/R}^{\infty} t^{k+2p-1}\left[f_p(x,t)-f(x)\right]\frac{\cos\left(tR-\frac{\pi}{2}(p+\delta+k/2)-\frac{\pi}{4}\right)}{(tR)^{p+\delta+(k+1)/2}}dt+$$

$$+CR^{k+2p}\int_{1/R}^{\infty} t^{k+2p-1}\left[f_p(x,t)-f(x)\right]O\left(\frac{1}{(tR)^{p+\delta+(k+3)/2}}\right)dt=N_1+N_2 .$$

Considering that $f(x)$ has a compact support in R_k, we represent the integral N_2 as:

$$N_2\le CR^{k/2+p-\delta-3/2}\int_{1/R}^{\rho_0} t^{k/2+p-\delta-5/2}\left|f_p(x,t)-f(x)\right|dt+$$

$$+CR^{k/2+p-\delta-3/2}\int_{\rho_0}^{\infty} t^{k/2+p-\delta-5/2}\left|f(x)\right|dt=T_1+T_2 .$$

Hence we have

$$T_2\le CR^{k/2+p-\delta-3/2}\int_{\rho_0}^{\infty} t^{k/2+p-\delta-5/2}\ dt\le CR^{k/2+p-\delta-3/2}=O\left(\frac{1}{R^{1+\eta}}\right)$$

uniformly with respect to $x\in G$.

For the integral T_1 we have

$$T_1\le CR^{k/2+p-\delta-3/2}\int_{1/R}^{\rho_0} t^{\alpha+p+k/2-\delta-5/2}\left(\int_\Sigma\frac{d\omega}{\Lambda^\sigma(x,t,\omega)}\right)dt .$$

By virtue of the inequality $\frac{1}{\lambda}+\frac{1}{\mu}=1$ we find

$$T_1\le CR^{k/2+p-\delta-3/2}\left(\int_{1/R}^{\rho_0} t^{(\alpha+k/2+p-\delta-5/2)\lambda}dt\right)^{1/\lambda}\left(\int_{1/R}^{\rho_0}\left(\int_\Sigma\frac{d\omega}{\Lambda^\sigma(x,t,\omega)}\right)^{\mu}dt\right)^{1/\mu}\le$$

$$\le CR^{k/2+p-\delta-3/2}\left(\int_{1/R}^{\rho_0} t^{(\alpha+k/2+p-\delta-5/2)\lambda}dt\right)^{1/\lambda}=CR^{k/2+p-\delta-3/2}R^{-(\alpha+k/2+p-\delta-5/2+1/\lambda)}=$$

$$=CR^{-\alpha+\eta}=O\left(\frac{1}{R^{\alpha-\eta}}\right).$$

Consequently,

$$N_2=O\left(\frac{1}{R^{\alpha-\eta}}\right)\ \text{ for }\ R\to\infty$$

uniformly with respect to $x \in G$.

It remains to estimate the integral N_1

To this end first of all estimate the integral

$$D = CR^{k/2+p-\delta-1/2}\left(\int_0^{1/R} t^{k/2+p-\delta-3/2+\alpha}\left(\int_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t,\omega)}\right)\right)dt \le$$

$$\le CR^{k/2+p-\delta-1/2}\left(\int_0^{1/R} t^{(\alpha+k/2+p-\delta-3/2)\lambda}dt\right)^{1/\lambda}\left(\int_0^{1/R}\left(\int_\Sigma \frac{d\omega}{\Lambda^\sigma(x,t,\omega)}\right)^\mu dt\right)^{1/\mu} \le$$

$$\le CR^{k/2+p-\delta-1/2}\left(\int_0^{1/R} t^{(\alpha+k/2+p-\delta-3/2)\lambda}dt\right)^{1/\lambda} = O\left(\frac{1}{R^{\alpha-\eta}}\right).$$

uniformly with respect to $x \in G$.

Considering the last remark instead of the integral N_1 it is sufficient to study the integral

$$N = CR^{k/2+p-\delta-1/2}\int_0^\infty \frac{f_p(x,t)-f(x)}{t^{\delta-p-k/2+3/2}}\cos\left[tR - \frac{\pi}{2}\left(p+\delta+\frac{k}{2}\right)-\frac{\pi}{4}\right]dt. \qquad (21)$$

Put

$$\psi_x(t) = \begin{cases} \varphi_x(t) & \text{for} \quad t \ge 0 \\ 0 & \text{for} \quad t < 0 \end{cases}.$$

Then instead of integral (21) it is sufficient to estimate the integral

$$R^{k/2+p-\delta-1/2}\int_{-\infty}^\infty \psi_x(t)e^{\pm itR}dt$$

for any value (+) and (-1).

By virtue of lemma 5 for the function $\psi_x(t)$ we have

$$\int_{-\infty}^\infty |\varphi_x(t+h)-\psi_x(t)|dt = O\left(h^{\alpha-2\eta}\right) \text{ for } h \to 0$$

uniformly with respect to $x \in G$, where $C \cap (\Phi \cup N(\Phi)) = \varnothing$.

For $h = \dfrac{\pi}{R}$ we have

$$\left|\int_{-\infty}^\infty \psi_x(t)e^{itR}dt\right| = \frac{1}{2\sin\pi/2}\left|\int_{-\infty}^\infty [\psi_x(t+h)-\psi_x(t)]e^{itR}dt\right| =$$

$$= \frac{1}{2}\left|\int_{-\infty}^\infty \left[\psi_x\left(t+\frac{\pi}{R}\right)-\psi_x(t)\right]e^{itR}dt\right| = O\left(R^{-\alpha+2\eta}\right) \quad \text{for} \quad R \to \infty$$

uniformly with respect to $x \in G$.

Consequently,

$$R^{k/2+p-\delta-1/2}\left|\int_{-\infty}^\infty \psi_x(t)e^{\mp itR}dt\right| = O\left(R^{k/2+p-\delta-1/2}R^{-\alpha+2\eta}\right) = O\left(R^{-\alpha+\eta}\right)$$

uniformly with respect to $x \in G$.

So we have proved, that

$$\tilde{S}_R^\delta(x,f)-2^{\frac{k-2}{2}}\Gamma\left(\frac{k}{2}\right)f(x)=O\left(\frac{1}{R^{\alpha-\eta}}\right) \qquad \eta<\frac{\alpha}{2}$$

uniformly with respect to $x\in G$.

In particular from this theorem it follows, that

$$\lim_{R\to\infty}\tilde{S}_R^\delta(x,f)=2^{\frac{k-2}{2}}\Gamma\left(\frac{k}{2}\right)f(x)$$

uniformly with respect to $x\in G$, where $C\cap(\Phi\cup N(\Phi))=\varnothing$. The theorem is proved.

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