1999 VOL. XI (XIX)

AMIROVA L.I.

ON THE SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR A CLASS OF SECOND ORDER OPERATOR-DIFFERENTIAL EQUATIONS

Abstract

Sufficient conditions for the existence of the solution to initial boundary value problem are found for second order operator-differential equations whose principal part contains a normal operator. These conditions are expressed by the coefficients of a operator-differential equation.

A theorem on the existence of holomorphic solutions of the initial-boundary value problem is proved for a class of the second order operator- differential equations whose symbol contains a normal operator at the principal part.

In a separable Hilbert space H consider the boundary-value problem

$$P(d/dz)u(z) = -u''(z) + A^2u(z) + A_1u'(z) + A_2u(z) = f(z), \quad z \in S_\alpha, \tag{1}$$

$$u'(0) = 0, \tag{2}$$

where $S_\alpha = \{z \,|\, |\arg z| < \alpha\}$, $0 < \alpha < \pi/2$, A_1, A_2, A are linear operators in H, $f(z)$ and $u(z)$ are holomorphic at the sector S_α vector functions with values in H.

Further, we assume the fulfillment of the following conditions:

1) A is a normal operator with compact continuous inverse A^{-1}, whose spectrum is contained in a cone sector

$$S_\varepsilon = \{\lambda \,|\, |\arg \lambda| \le \varepsilon\}, \quad 0 \le \varepsilon < \pi/2;$$

2) $B_j = A_j A^{-j}$ $(j = 1,2)$ are bounded operators in H;

3) the members α and ε satisfy the condition: $0 < \alpha + \varepsilon < \pi/2$.

It is obvious that by fulfilling the condition 1) the operator A has a polar expansion $A = UC$, where U is a unitary, and C is a positive-defined self-adjoint operator in H, moreover, $D(A) = D(A^*) = D(C)$ and for any $x \in D(A)$ $\|Ax\| = \|A^*x\| = \|Cx\|$. Definition domain of the operator C^γ $(\gamma \ge 0)$ becomes the Hilbert space H_γ with respect to the norm $\|x\|_\gamma = \|C^\gamma x\|, x \in D(C^\gamma)$. Further, denote by $L_2(R_+ : H)$ a Hilbert space of vector-functions $f(t)$ determined in $R_+ = (0,\infty)$ with values in H, for which

$$\|f\|_{L_2} = \left(\int_0^\infty \|f(t)\|^2 dt\right)^{\frac{1}{2}} < \infty.$$

Denote by $H_2(\alpha : H)$ a set of vector-functions $f(z)$ determined in S_α, with values in H, which are holomorphic in S_α and at each $\varphi \in [-\alpha, \alpha]$ of vector functions $f_\varphi = f(te^{i\varphi}) \in L_2(R_+ : H)$. This linear set is a Hilbert space with respect to the norm [1]

$$\|f\|_\alpha = \frac{1}{\sqrt{2}}\left(\|f_\alpha\|_{L_2}^2 + \|f_{-\alpha}\|_{L_2}^2\right)^{\frac{1}{2}}.$$

Determine the following Hilbert spaces

$$W_2^2(\alpha:H)=\{u(z)|u''(z)\in H_2(\alpha:H),\ A^2u\in H_2(\alpha:H)$$

$$\overset{\circ}{W}{}_2^2(\alpha:H:1)=\{u(z)|u(z)\in W_2^2(\alpha:H),\ u'(0)=0\}$$

with the norm

$$|||u|||_\alpha=\left(\|u''\|_\alpha^2+\|A^2u\|_\alpha^2\right)^{\frac{1}{2}}.$$

Definition 1. *The function* $u(z)\in W_2^2(\alpha:H)$ *we call the regular solution of the problem (1), (2), if it satisfies the equation (1) identically in* S_α, *and the condition (2) is fulfilled in the sense*

$$\lim_{\substack{z\to 0\\ |\arg z|<\alpha}}\|u'(z)\|_{H_{1/2}}=0.$$

Definition 2. *The problem (1), (2) we call regular solvable, if for any* $f(z)\in H_2(\alpha:H)$ *it has a regular solution* $u(z)$, *moreover*

$$|||u(z)|||_\alpha\le const\|f\|_\alpha.$$

Determine the following operators on the space $\overset{\circ}{W}{}_2^2(\alpha:H:1)$:

$\mathcal{P}_0u=P_0(d/dz)u=-u''(z)+A^2u(z),\ \mathcal{P}_1=P_1(d/dz)u=A_1u'(z)+A_2u(z),\qquad \mathcal{P}u=\mathcal{P}_0u+\mathcal{P}_1u,$

$u\in\overset{\circ}{W}{}_2^2(\alpha:H:1)$.

First we prove some lemmas.

Lemma 1. *Let the condition (1) be fulfilled, then the mapping*

$$e^{-zA}:H_{3/2}\to W_2^2(\alpha:H).$$

Proof. Let $x\in H_{3/2}$. They by definition

$$|||e^{-zA}|||_\alpha^2=2\|A^2e^{-zA}x\|_\alpha=\left\|A^2e^{-te^{i\alpha}}x\right\|_{L_2}^2+\left\|A^2e^{-te^{-i\alpha}}x\right\|_{L_2}^2. \tag{3}$$

If $\{\lambda_n\}_{n=1}^\infty$ are eigen-values, and $\{e_n\}_{n=1}^\infty$ are corresponding eigen-vector of A, then

$$\left\|A^2e^{-te^{\pm i\alpha}}x\right\|_{L_2}^2=\int_0^\infty\left\|\sum_{n=1}^\infty\lambda_n^2e^{-\lambda_nte^{\pm i\alpha}}(x,e_n)e_n\right\|_H^2dt=$$

$$=\int_0^\infty\sum_{n=1}^\infty|\lambda_n|^4e^{-2t|\lambda_n|\cos(\pm\alpha+\arg\lambda_n)}|(x,e_n)|^2dt\le\int_0^\infty\sum_{n=1}^\infty|\lambda_n|^4e^{-2t|\lambda_n|\cos(\alpha+\varepsilon)}|(x,e_n)|^2dt=$$

$$=\sum_{n=1}^\infty|\lambda_n|^3[2\cos(\alpha+\varepsilon)]^{-1}|(x,e_n)|=[2\cos(\alpha+\varepsilon)]^{-1}\|x\|_{3/2}.$$

Thus, we get from the equality (3)

$$|||e^{-zA}x|||_\alpha\le[\cos(\alpha+\varepsilon)]^{-1/2}\|x\|_{3/2}.$$

The Lemma is proved.

Lemma 2. *The operator* $\mathcal{P}_0$ *maps the space* $\overset{\circ}{W}{}_2^2(0:H:1)$ *onto* $H_2(\alpha:H)$ *isomorphically.*

Proof. It is obvious that the equation $\mathcal{P}_0u=-u''+A^2u=0$ has a zero solution from the space $\overset{\circ}{W}{}_2^2(\alpha:H:1)$. On the other hand, it is easy to see that (see [1]) for any $f(z)\in H_2(\alpha:H)$ the vector function

$$\vartheta(z)=\frac{1}{2\pi i}\left(\int_{\Gamma_{\pi/2+\alpha}}\left(-\lambda^2E+A^2\right)^{-1}\hat{f}(\lambda)e^{\lambda z}d\lambda-\int_{\Gamma_{-(\pi/2+\alpha)}}\left(-\lambda^2E+A^2\right)^{-1}\hat{f}(\lambda)e^{\lambda z}d\lambda\right)$$

belongs to the space $W_2^2(\alpha:H)$ and satisfies the equality $P_0u=f$, where $\Gamma_{\pm(\pi/2+\alpha)}=$ $=\{\lambda|\arg\lambda=\pm(\pi/2+\alpha)\}$, $\hat{f}(\lambda)$ is a Laplace transformation of the vector function $f(z)$. Then the equation $P_0u=f$ has a general solution in the form:

$$u(z)=\vartheta(z)+e^{-zA}\left(A^{-1}\vartheta'(0)\right).$$

Since $\vartheta(z)\in W_2^2(\alpha:H)$, according to the trace theorem it follows that $\vartheta'(0)\in H_{1/2}$. Then $A^{-1}(\vartheta'(0))\in H_{3/2}$. Therefore by Lemma 1, $e^{-zA}\left(A^{-1}\vartheta'(0)\right)\in W_2^2(\alpha:H)$. It is obvious that $u(z)$ satisfies the boundary condition (2). Thus, the equation $P_0u=f$ has a solution from $\overset{\circ}{W}{}_2^2(\alpha:H:1)$ for any $f(z)\in H_2(\alpha:H)$. Since $\|P_0u\|_\alpha\le\sqrt{2}|\!|\!|u|\!|\!|_\alpha$, the affirmation of the Lemma follows from the Banach's inverse operator theorem.

Lemma 3. *For any $u(z)\in\overset{\circ}{W}{}_2^2(\alpha:H:1)$ it holds the inequality*

$$\|P_0u\|_\alpha^2\ge|\!|\!|u|\!|\!|_\alpha^2+2\cos2(\alpha+\varepsilon)\|Au'\|_\alpha^2. \tag{4}$$

Proof. Let $u(z)\in\overset{\circ}{W}{}_2^2(\alpha:H:1)$, then the vector-function $P_0(d/dz)u(z)=-u''(z)+$ $+A^2u(z)$ has on boundaries $\Gamma_{\pm(\pi/2+\alpha)}$ the boundary values

$$\psi_{\pm\alpha}(t)=-u''_{\pm\alpha}(t)e^{\mp2i\alpha}+A^2u_{\pm\alpha}(t).$$

By definition

$$\|P_0u\|_\alpha^2=\frac{1}{2}\left(\|\psi_\alpha\|_{L_2}^2+\|\psi_{-\alpha}\|_{L_2}^2\right)=|\!|\!|u|\!|\!|_\alpha^2-\mathrm{Re}\left[\left(u''_\alpha e^{-2i\alpha},A^2u_\alpha\right)_{L_2}+\left(-u''_{-\alpha}e^{2i\alpha},A^2u_{-\alpha}\right)_{L_2}\right]. \tag{5}$$

By considering that $u'_{\pm\alpha}(0)=0$, then after integrating by parts, we have

$$-\mathrm{Re}\left(u''_{\pm\alpha}e^{\mp2i\alpha},A^2u_{\pm\alpha}\right)_{L_2}=\mathrm{Re}\,e^{\mp2i\alpha}\left(A^*u'_{\pm\alpha},Au'_{\pm\alpha}\right)_{L_2}. \tag{6}$$

Since for any $x\in D(A)$,

$$\mathrm{Re}\,e^{\mp2i\alpha}\left(A^*x,Ax\right)=\mathrm{Re}\sum_{n=1}^{\infty}e^{\mp2i\alpha}\lambda_n^2|(x,e_n)|^2=\mathrm{Re}\sum_{n=1}^{\infty}e^{\mp2i\alpha}|\lambda_n|^2e^{2i\arg\lambda_n}|(x,e_n)|^2=$$

$$=\sum_{n=1}^{\infty}|\lambda_n|^2|(x,e_n)|^2\cos2(\mp\alpha+\arg\lambda_n)\ge\cos2(\alpha+\varepsilon)\sum_{n=1}^{\infty}|\lambda_n|^2|(x,e_n)|^2=\cos2(\alpha+\varepsilon)\|Ax\|^2,$$

from the equality (5) by considering (6) we get

$$\|P_0u\|_\alpha^2\ge|\!|\!|u|\!|\!|_\alpha^2+\cos2(\alpha+\varepsilon)\left[\|Au'_\alpha\|_{L_2}^2+\|Au'_{-\alpha}\|_{L_2}^2\right]=|\!|\!|u|\!|\!|_\alpha^2+2\cos2(\alpha+\varepsilon)\|Au'\|_\alpha.$$

The Lemma is proved.

Now prove the principal theorem.

Theorem 1. *Let the conditions (1), (2) and (3) be fulfilled, moreover, it holds the inequality*

$$K(\varepsilon,\alpha)=C_1(\varepsilon,\alpha)\|B_1\|+C_2(\varepsilon,\alpha)\|B_2\|<1,$$

where

$$C_1(\varepsilon,\alpha)=[2\cos(\alpha+\varepsilon)]^{-1},$$

$$C_2(\varepsilon,\alpha)=\begin{cases}1, & for\quad 0<\alpha+\varepsilon\le\pi/4\\ \left[\sqrt{2}\cos(\alpha+\varepsilon)\right]^{-1}, & for\quad \pi/4\le\alpha<\pi/2\end{cases}$$

Then the problem (1), (2) is regulary solvable.

Proof. After substituting $\mathcal{P}_0u=g$ we get that the equation $\mathcal{P}u=\mathcal{P}_0u+\mathcal{P}_1u=f$, $u\in\overset{\circ}{W}{}_2^2(\alpha:H:1)$, $f\in H_2(\alpha:H)$, turns into the equation $\vartheta+\mathcal{P}_1\mathcal{P}_0^{-1}\vartheta=f$ on the space $H_2(\alpha:H)$. Since for any $g\in H_2(\alpha:H)$

$$\left\|\mathcal{P}_1\mathcal{P}_0^{-1}g\right\|_\alpha=\|\mathcal{P}_1u\|_\alpha=\|A_1u'+A_2u\|_\alpha\le\|B_1\|\cdot\|Au'\|_\alpha+\|B_2\|\cdot\left\|A^2u\right\|_\alpha, \tag{7}$$

then we must estimate the norms $\|Au'\|_\alpha$ and $\left\|A^2u\right\|_\alpha$.

By definition

$$\|Au'\|_\alpha^2=\|Cu'\|_\alpha^2=\frac{1}{2}\left(\|Cu_{+\alpha}\|_{L_2}^2+\|Cu_{-\alpha}\|_{L_2}^2\right). \tag{8}$$

Considering that $u'_{\pm\alpha}(0)=0$, after integrating by parts we get

$$\|C_{\pm\alpha}u'\|_{L_2}^2=\int_0^\infty\left(Cu'_{\pm\alpha},Cu'_{\pm\alpha}\right)dt=-\left(C^{1/2}u_{\pm\alpha}(0),C^{3/2}u_{\pm\alpha}u'(0)\right)-\int_0^\infty\left(u''_{\pm\alpha},C^2u_{\pm\alpha}\right)dt\le$$

$$\le\|u''_{\pm\alpha}\|_{L_2}\left\|C^2u_{\pm\alpha}\right\|_{L_2}\le\frac{1}{2}\left(\|u''_{\pm\alpha}\|_{L_2}^2+\left\|C^2u\right\|_{L_2}^2\right).$$

Then it follows from the equality (8), that

$$\|Au'\|_\alpha^2\le\frac{1}{2}\left(\|u''_\alpha\|_{L_2}^2+\left\|A^2u_\alpha\right\|_{L_2}^2+\|u''_{-\alpha}\|_{L_2}^2+\left\|C^2u_{-\alpha}\right\|_{L_2}^2\right)$$

or

$$\|Au'\|_\alpha^2\le\frac{1}{2}\|u\|_\alpha^2. \tag{9}$$

Taking into account the last inequality in the equality (4), we get

$$\|Au'\|_\alpha^2\le2^{-1}\|\mathcal{P}_0u\|_\alpha^2-\cos2(\alpha+\varepsilon)\|Au'\|_\alpha^2.$$

Hence we obtain

$$\|Au'\|_\alpha\le[2\cos(\alpha+\varepsilon)]^{-1}\|\mathcal{P}_0u\|_\alpha=C_1(\varepsilon,\alpha)\|g\|_\alpha. \tag{10}$$

To estimate the second summand in the equality (7) note that for $0<\alpha+\varepsilon\le\frac{\pi}{4}$ $\cos2(\alpha+\varepsilon)\ge0$, therefore from the inequality (4) it follows

$$\left\|A^2u\right\|_\alpha\le\|\mathcal{P}_0u\|_\alpha=\|g\|_\alpha, \tag{11}$$

and for $\frac{\pi}{4}\le\alpha+\varepsilon<\frac{\pi}{2}$ from the inequality (4) by considering (9) it follows that

$$\|\mathcal{P}_0u\|_\alpha^2\ge[1+\cos(\alpha+\varepsilon)]\|u\|_\alpha^2\ge2\cos^2(\alpha+\varepsilon)\left\|A^2u\right\|_\alpha^2,$$

i.e.

$$\left\|A^2u\right\|_\alpha\le\left[\sqrt{2}\cos(\alpha+\varepsilon)\right]^{-1}\|g\|_\alpha\quad\left(\frac{\pi}{4}\le\alpha+\varepsilon<\frac{\pi}{2}\right). \tag{12}$$

We get from the inequalities (11) and (12)

$$\left\|A^2u\right\|_\alpha \le C_2(\varepsilon,\alpha)\|g\|_\alpha . \qquad (13)$$

Taking into account the inequalities (10) and (13) in the estimate (7) we get

$$\left\|\mathcal{P}_1\mathcal{P}_0^{-1}g\right\|_\alpha \le \left(C_1(\varepsilon,\alpha)\|B_1\| + C_2(\varepsilon,\alpha)\|B_2\|\right)\|g\|_\alpha ,$$

i.e. the norm of the operator $\mathcal{P}_1\mathcal{P}_0^{-1}$ by the condition of Theorem is less than $K(\varepsilon,\alpha)<1$. Therefore, the operator $E+\mathcal{P}_1\mathcal{P}_0^{-1}$ converts to the space $H_2(\alpha:H)$. Therefore, $g=\left(E+\mathcal{P}_1\mathcal{P}_0^{-1}\right)f$, and $u=\mathcal{P}_0^{-1}\left(E+\mathcal{P}_1\mathcal{P}_0^{-1}\right)f$.

Hence, it follows that

$$\|u\|_\alpha \le \left\|\mathcal{P}_0^{-1}\right\|\cdot\left\|\left(E+\mathcal{P}_1\,\mathcal{P}_0^{-1}\right)^{-1}\right\|\cdot\|f\|_\alpha \le \frac{\left\|\mathcal{P}_0^{-1}\right\|}{1-K(\varepsilon,\alpha)}\|f\|_\alpha = const\|f\|_\alpha .$$

The Theorem is proved.

Note that the problem (1), (2) for $\alpha=0$ and $\varepsilon=0$, in particular was considered in paper [2]. The boundary value problem for the equation (1) with a boundary condition $u(0)=0$ has been investigated by many authors (see for instance [1-5]) in different situations.

The author expresses her gratitude to prof. S.S.Mirzoyev and ass. Dosent M.M. Huseynov for their great attention to the work.

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Amirova L.I.
Baku State University named after M. Rasulzadeh.
23, Z.I. Khalilov str., 370148, Baku, Azerbaijan.

Received November 23, 1999; Revised February 9, 2000.
Translated by Aliyeva E.T.