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THE INVERSE THEOREM IN THE APPROXIMATION OF PRODUCTS BY SUMS

Abstract

The problem of approximation of the functions of several variables by the sums of functions depending on the groups of fewer variables is investigated. Existance of the solution of one functional equation system connected with the structure of approximated functions is established.

In [1] the best approximating element in the approximation of the product of functions by sums of the functions of less number of variables was constructed. In the present work it is established that this method of construction of the extremal function of the best approximation reduces to existence of the solution of some system of functional equations connected with the structure of the approximated function.

Let $t=(t_1,...,t_m)$, where

$$t_i\left(x_{k_{i-1}+1},x_{k_{i-1}+2},...,x_{k_i}\right),\ \ i=\overline{1,m};\quad 0=k_0<k_1<\cdots<k_m=n$$

represent some division of a set of variables $t(x_1,...,x_n)$ into m groups. Let us consider a real function $f=f(t)=f(t_1,...,t_m)$ determined in the parallelepiped $[a_1,b_1;\cdots;a_n,b_n]$, which we will denote by $T_m=[c_1,d_1;\cdots;c_m,d_m]$, where $c_i=\left(a_{k_{i-1}+1},...,a_{k_i}\right)$, $d_i=\left(b_{k_{i-1}+1},...,b_{k_i}\right)$, $i=\overline{1,m}$.

Let's denote the best uniform approximation of function f by sums $\sum_{\nu=1}^{m}\varphi_\nu$, function of $m-1$ groups of variable $\varphi_\nu=\varphi_\nu(t_1,...,t_{\nu-1},t_{\nu+1},...,t_m)=\varphi_\nu(t\setminus t_\nu)$

$$E\left[f,\Sigma_1^m\varphi_\nu,T_m\right]=\inf_{\Sigma_1^m\varphi_\nu}\sup_{p\in T_m}\left|\left[f-\Sigma_1^m\varphi_\nu\right](p)\right|,$$

where the below side are spread for the class of all bounded real functions of view $\Sigma_1^m\varphi_\nu$. The function for which this below side is reached is called the best approximating function.

We will say that function f increases $f\uparrow$ (decreases $f\downarrow$), if it is an increasing (decreasing) one by every variable. Let's denote by ПП the class of functions of view $f=\prod_{\nu=1}^{m}f_\nu(t_\nu)$, for which even number (or zero) of functions are $f_\nu\downarrow$, and the rest of them are $\uparrow$, $\nu=\overline{1,m}$.

Theorem. *Let function* $f=\prod_{\nu=1}^{m}f_\nu(t_\nu)$ *increases and the sum*

$$\Sigma_f^0\overset{df}{=}\sum_{\nu=1}^{m}\varphi_\nu^0(t\setminus t_\nu)=f(t)-\prod_{\nu=1}^{m}\left[f_\nu(t_\nu)-f_\nu(\bar t_\nu)\right]$$

is the best approximating function for f, *where* $\bar t=(\bar t_1,...,\bar t_m)$ *is some point from the determination domain* T_m. *Then* $\bar t$ *is the solution of the system of equations*

$$f_\nu(t_\nu)=\frac{1}{2}\left[f_\nu(c_\nu)+f_\nu(d_\nu)\right],\ \ \nu=\overline{1,m}. \tag{1}$$

Proof. We will need a row of auxiliary results. Let $M=\{1,\dots,m\}$, $I,J\subset M$.

Lemma 1. *The correlations are valid*

a) $I\setminus J=I\setminus(I\cap J)$;

b) $J\setminus I=(M\setminus I)\setminus[(M\setminus I)\cap(M\setminus J)]$;

c) $I\setminus(I\setminus J)=I\cap J$;

d) $(M\setminus I)\setminus(J\setminus I)=(M\setminus I)\cap(M\setminus J)$;

e) $M=(I\setminus J)\cup(J\setminus I)\cup[(M\setminus I)\cap(M\setminus J)]$;

f) *if the number of elements of the sets I and J of different evenness then* $mes(I\setminus J)\neq mes(J\setminus I)$.

All these correlations are easily proved. For example let us prove d) and f).

Proof of d).

$$x\in(M\setminus I)\setminus(J\setminus I)\Rightarrow\begin{cases}x\in M\setminus I\\ x\notin J\setminus I\Rightarrow x\notin J\Rightarrow x\in M\setminus J\Rightarrow x\in(M\setminus I)\cap(M\setminus J)\end{cases};$$

$$x\in(M\setminus I)\cap(M\setminus J)\Rightarrow\begin{cases}x\in M\setminus I\\ x\notin M\setminus J\Rightarrow x\notin J\Rightarrow x\notin J\setminus I\Rightarrow x\in(M\setminus I)\setminus(J\setminus I)\end{cases}.$$

Proof of f). If $I\cap J=\varnothing$, then $mes(I\setminus J)=mesI$, $mes(J\setminus I)=mesJ$ and f) is valid. But if $I\cap J\neq\varnothing$, then denoting $mesI\cap J=r\neq 0$ we will obtain $mesI\setminus J=mesI-r$, $mes(J\setminus I)=mesJ-r$ that again reduces to validity of f).

Let's consider the expression

$$L_m(f,Q)\overset{df}{=}\sum_{i=1}^{2^m}(-1)^{\delta_i}f(Q_i),$$

where Q_i are all possible tops of the "parallelepiped"

$$Q=[t_1',t_1'';\dots;t_m',t_m'']\subset T_m,$$

δ_i is the number t_i', $i=\overline{1,m}$ which are the coordinates of point Q_i, $t_i'\le t_i''$ means $x_j'\le x_j''$, $x_j\in t_i$.

Let's determine the function

$$g=g(t)=L_m(f,K),\quad K=[c_1,t_1;\dots;c_m,t_m].$$

It is not difficult to be persuaded that for a function of the view $f=\prod_{\nu=1}^{m}f_\nu(t_\nu)$ the function g can be represented in the form

$$g=\prod_{\nu=1}^{m}[f_\nu(t_\nu)-f_\nu(c_\nu)] \tag{2}$$

and so $g(t_1,\dots,c_\nu,\dots,tm)=0$, $\nu=\overline{1,m}$.

Let $\bar t=(\bar t_1,\dots,\bar t_m)\in T_m$ be the point from the formulation of the theorem .

Let's determine the function

$$R(t)=L_m(g,D),\quad D=[\bar t_1,t_1;\dots;\bar t_m,t_m]. \tag{3}$$

Let's denote

$$g_\nu(t_\nu)=f_\nu(t_\nu)-f_\nu(c_\nu),\quad \nu=\overline{1,m}. \tag{4}$$

It is easy to note that for a function of the view $f=\prod_{\nu=1}^{m}f_\nu(t_\nu)$ the function R can be represented in the form

$$R(t)=\prod_{\nu=1}^{m}\left[g_\nu(t_\nu)-g_\nu(\bar{t}_\nu)\right]. \tag{5}$$

According to the representation (2) the function g distinct from function f for the sum of the view $\sum_{\nu=1}^{m}\varphi_\nu(t\setminus t_\nu)$ and according to (3) function R distinct from function g for the function of the same view:

$$g=f-\sum_{\nu=1}^{m}\varphi_\nu^*(t\setminus t_\nu),\quad R=g-\sum_{\nu=1}^{m}\varphi_\nu^{**}(t\setminus t_\nu),$$

where

$$\Sigma^*\overset{df}{=}\sum_{\nu=1}^{m}\varphi_\nu^*(t\setminus t_\nu)\ \text{ and }\ \Sigma^{**}\overset{df}{=}\sum_{\nu=1}^{m}\varphi_\nu^{**}(t\setminus t_\nu)$$

are the concrete sums. So

$$E[R,\Sigma\varphi_\nu]=E[g,\Sigma\varphi_\nu]=E[f,\Sigma\varphi_\nu].$$

Further by virtue of (2) and (4) $g=\prod_{\nu=1}^{m}g_\nu(t_\nu)$ and according to the condition of theorem the functions g_ν, $\nu=\overline{1,m}$ are increasing ones. Then by virtue of the representation (5) one can confirm that function R gets its maximal and minimal values only on the tops of the "parallelepiped" T_m. Moreover, $\sup\limits_{t\in T_m}R(t)$ can be reached in those tops where even (or zero) number from the groups of variables t_ν gets values c_ν, and the others get $-d_\nu$ (in these points $R\geq 0$), $\inf\limits_{t\in T_m}R(t)$ can be reached in those tops T_m, where odd number t_ν gets values c_ν and other get $-d_\nu$.

The above-said can be written in the form

$$\left.\begin{aligned}\sup_{T_m}R(t)&=\max\left\{R\left(c_{\alpha_1},\dots,c_{\alpha_{2p}},d_{\alpha_{2p+1}},\dots,d_{\alpha_m}\right)\right\}\\ \inf_{T_m}R(t)&=\min\left\{R\left(c_{\beta_1},\dots,c_{\beta_{2q-1}},d_{\beta_{2q}},\dots,d_{\beta_m}\right)\right\}\end{aligned}\right\} \tag{6}$$

Here $(\alpha_1,\dots,\alpha_m)$, $(\beta_1,\dots,\beta_m)$ are arbitrary transpositions of numbers $1,\dots,m$, and $R(t_{\alpha_1},\dots,t_{\alpha_m})$, $R(t_{\beta_1},\dots,t_{\beta_m})$ is the conditional note of the value of function R in the corresponding points; $1\leq p\leq\left[\frac{m}{2}\right]$, $1\leq q\leq\left[\frac{m+1}{2}\right]$, $[a]$ is the integer part a.

It is clear that max and min in (6) are achieved. Let's denote by $I=\{\alpha_1^0,\dots,\alpha_{2p_0}^0\}$ the set of indices from M which sup is achieved for and by $J=\{\beta_1^0,\dots,\beta_{2q^0-1}^0\}$ the set of indices which inf is archived for in (6). Here p_0 and q_0 are the fixed numbers; it is obvious $I,J\subset M=\{1,\dots,m\}$. According to (6) we have

$$\left.\begin{aligned}R\left(c_{\alpha_1^0},\dots,c_{\alpha_{2p_0}^0},d_{\alpha_{2p_0+1}},\dots,d_{\alpha_m}\right)&\geq R\left(c_{\alpha_1},\dots,c_{\alpha_{2p}},d_{\alpha_{2p+1}},\dots,d_{\alpha_m}\right)\\ R\left(c_{\beta_1^0},\dots,c_{\beta_{2q^0-1}^0},d_{\beta_{2q^0}},\dots,d_{\beta_m}\right)&\leq R\left(c_{\beta_1},\dots,c_{\beta_{2q-1}},d_{\beta_{2q}},\dots,d_{\beta_m}\right)\end{aligned}\right\} \tag{7}$$

We will need the following confirmation.

Theorem A ([1], p.18). *Let* $f=\prod_{\nu=1}^{m} f_\nu(t_\nu)\in \Pi\Pi$ *and each of functions* $f_\nu(t_\nu)$, $\nu=\overline{1,m}$ *increases and decreases.*

Then $E\left[f,\sum_1^m \varphi_\nu(t\setminus t_\nu)\right]=2^{-m}\prod_{\nu=1}^{m}|f_\nu(d_\nu)-f_\nu(c_\nu)|$.

As it was above mentioned the functions $g_\nu(t_\nu)$, $\nu=\overline{1,m}$ increase and $g=\prod_{\nu=1}^{m} g_\nu(t_\nu)$. Then using Theorem A the best approximation of function g by diversification of functions of the form $\sum_{\nu=1}^{m}\varphi_\nu(t\setminus t_\nu)$ can be calculated by the formula

$$E\left[g,\sum_1^m \varphi_\nu(t\setminus t_\nu)\right]=2^{-m}\prod_{\nu=1}^{m}[g_\nu(d_\nu)-g_\nu(c_\nu)]$$

and as far as by (4) $g_\nu(c_\nu)=0$, $\nu=\overline{1,m}$, then

$$E\left[g,\sum_1^m \varphi_\nu\right]=2^{-m}\prod_{\nu=1}^{m} g_\nu(d_\nu). \tag{8}$$

By the condition of the theorem the sum $\sum_f^0$ is the best approximation for f and by definition of functions g and R have the form $g=f-\Sigma^*$, $R=g-\Sigma^{**}$, where Σ^* and Σ^{**} are the concrete sums of the form $\sum_{\nu=1}^{m}\varphi_\nu(t\setminus t_\nu)$. Then it is not difficult to be certain in that the functions $\Sigma_g^0\overset{df}{=}\Sigma_f^0-\Sigma^*$ and $\Sigma_R^0\overset{df}{=}\Sigma_g^0-\Sigma^{**}$ are correspondingly best approximations for g and R.

Let show it for example for function g. We have

$$E_f=\inf_{\sum\varphi_\nu}\|f-\Sigma\varphi_\nu\|=\|f-\Sigma_f^0\|=\|f-\Sigma^*+\Sigma^*-\Sigma_f^0\|=\|g-(\Sigma_g^0-\Sigma^*)\|.$$

Taking it into account we obtain

$$E_f=\inf_{\sum\varphi_\nu}\|f-\Sigma\varphi_\nu\|=\inf_{\sum\varphi_\nu}\|f-\Sigma^*+\Sigma^*-\Sigma\varphi_\nu\|=$$

$$=\inf_{\sum\varphi_\nu}\|g-(\Sigma\varphi_\nu-\Sigma^*)\|=\inf_{\sum\varphi_\nu}\|g-\Sigma\varphi_\nu\|=\|g-(\Sigma_f^0-\Sigma^*)\|,$$

that's the sum $\Sigma_g^0=\Sigma_f^0-\Sigma^*$ is the best approximation for g. Let us show that $R=g-\Sigma_g^0$. As far as $R=g-\Sigma^{**}$, then for that it is sufficient to be certain in that $\Sigma_g^0=\Sigma^{**}$. Let's do it. By virtue of the representation of function g in (2) we have

$$g=\prod_{\nu=1}^{m}[f_\nu(t_\nu)-f_\nu(c_\nu)]=f-\left\{f-\prod_1^m[f_\nu(t_\nu)-f_\nu(c_\nu)]\right\}=f-\Sigma^*,$$

whence

$$\Sigma^*=f-\prod_{\nu=1}^{m}[f_\nu(t_\nu)-f_\nu(c_\nu)].$$

The above-said the conditions of the theorem let write

$$\Sigma_g^0 = \Sigma_f^0 - \Sigma^* = \left\{ f - \prod_1^m [f_\nu(t_\nu) - f_\nu(\bar{t}_\nu)] \right\} - \left\{ f - \prod_1^m [f_\nu(t_\nu) - f_\nu(c_\nu)] \right\} = \\ = \prod_1^m [f_\nu(t_\nu) - f_\nu(c_\nu)] - \prod_1^m [f_\nu(t_\nu) - f_\nu(\bar{t}_\nu)] . \tag{9}$$

Further,

$$R = \prod_1^m [g_\nu(t_\nu) - g_\nu(\bar{t}_\nu)] = g - \left\{ g - \prod_1^m [g_\nu(t_\nu) - g_\nu(\bar{t}_\nu)] \right\} = g - \Sigma^{**},$$

whence and by virtue of (9)

$$\Sigma^{**} = \prod_1^m [f_\nu(t_\nu) - f_\nu(c_\nu)] - \prod_1^m [g_\nu(t_\nu) - g_\nu(\bar{t}_\nu)] = \Sigma_g^0 .$$

Therefore, $R = g - \Sigma_g^0$ and so

$$\|R\| = E_g = 2^{-m} \prod_{\nu=1}^m g_\nu(d_\nu). \tag{10}$$

Comparing (10) with (6) we obtain

$$\sup_{T_m} R = R\left(c_{\alpha_1^0}, \ldots, c_{\alpha_{2p_0}^0}, d_{\alpha_{2p_0+1}}, \ldots, d_{\alpha_{2m}} \right) = 2^{-m} \prod_1^m g_\nu(d_\nu). \tag{11}$$

As for as the set $\Sigma = \left\{ \sum_{\nu=1}^m \varphi_\nu (t \setminus t_\nu) \right\}$ has the property: for an arbitrary constant c

$$\sum \varphi_\nu \in \Rightarrow c \sum \varphi_\nu \in \Sigma,$$

then (see [1], p.26) for the difference $g - \Sigma_g^0$ it takes place the equality

$$\sup_{T_m} \{ g - \Sigma_g^0 \} = - \inf_{T_m} \{ g - \Sigma_g^0 \}$$

or

$$\sup_{T_m} R = - \inf_{T_m} R ,$$

whence by (5) and (11)

$$R\left(c_{\beta_1^0}, \ldots, c_{\beta_{2q_0-1}^0}, d_{\beta_{2q_0}}, \ldots, d_{\beta_m} \right) = -2^{-m} \prod_{\nu=1}^m g_\nu(d_\nu). \tag{12}$$

Using the denotations $I = \{ \alpha_1^0, \ldots, \alpha_{2p_0}^0 \}$ and $J = \{ \beta_1^0, \ldots, \beta_{2q_0-1}^0 \}$, and also the representation of function R from (5) the correlations (11) and (12) can be written in the form

$$\prod_{i \in I} [g_i(c_i) - g_i(\bar{t}_i)] \prod_{k \in M \setminus I} [g_k(d_k) - g_k(\bar{t}_k)] = 2^{-m} \prod_1^m g_\nu(d_\nu),$$

$$\prod_{i \in J} [g_i(c_i) - g_i(\bar{t}_i)] \prod_{k \in M \setminus J} [g_k(d_k) - g_k(\bar{t}_k)] = -2^{-m} \prod_1^m g_\nu(d_\nu).$$

But $g_i(c_i) = 0$, $i \in M$, moreover, as it was above mentioned, the set I contains the even number and the set J contains the odd number elements, so these equalities can be written in the following form

$$\prod_{i \in I} g_i(\bar{t}_i) \prod_{k \in M \setminus I} [g_k(d_k) - g_k(\bar{t}_k)] = 2^{-m} \prod_1^m g_\nu(d_\nu), \tag{13}$$

$$\prod_{i\in J} g_i(\bar t_i) \prod_{k\in M\setminus J}[g_k(d_k)-g_k(\bar t_k)] = 2^{-m}\prod_{1}^{m} g_v(d_v). \tag{14}$$

The correlations (7), (11)-(14) let write

$$\prod_{i\in I} g_i(\bar t_i) \prod_{k\in M\setminus I}[g_k(d_k)-g_k(\bar t_k)] \ge \prod_{i\in \mathcal{J}} g_i(\bar t_i) \prod_{k\in M\setminus \mathcal{J}}[g_k(d_k)-g_k(\bar t_k)], \tag{15}$$

$$\prod_{i\in J} g_i(\bar t_i) \prod_{k\in M\setminus J}[g_k(d_k)-g_k(\bar t_k)] \ge \prod_{i\in Ж} g_i(\bar t_i) \prod_{k\in M\setminus Ж}[g_k(d_k)-g_k(\bar t_k)], \tag{16}$$

where $\mathcal{J}$ and $Ж$ are an arbitrary subsets M. Let's denote by a_p the set containing P elements.

Lemma 2. *Let A mean one the subsebts $I_{2r}\subset I$ or $J_{2r}\subset J$, and B mean one of the subsets $I^*_{2r}\subset M\setminus I$ or $J^*_{2s}\subset M\setminus J$. The inequalities are valid:*

$$\prod_{i\in A} g_i(\bar t_i) \ge \prod_{i\in A}[g_i(d_i)-g_i(\bar t_i)], \tag{17}$$

$$\prod_{k\in B}[g_k(d_k)-g_k(\bar t_k)] \ge \prod_{k\in B} g_k(\bar t_k). \tag{18}$$

Proof. Let I_{2r} be some subset I containing an even number of elements. As $\mathcal{J}$ we take $\mathcal{J}=I\setminus I_{2r}$ and write the inequality (15)

$$\prod_{i\in I} g_i(\bar t_i) \prod_{k\in M\setminus I}[g_k(d_k)-g_k(\bar t_k)] \ge \prod_{i\in I\setminus I_{2r}} g_i(\bar t_i) \prod_{k\in M\setminus (I\setminus I_{2r})}[g_k(d_k)-g_k(\bar t_k)].$$

Abbreviating by cancellation the same products from both parts of this inequality (they are positive) we will obtain (17). If $A=J_{2s}$, then it should take $Ж=J\setminus J_{2s}$ containig on odd number of elements and write (16)

$$\prod_{i\in J} g_i(\bar t_i) \prod_{k\in M\setminus J}[g_k(d_k)-g_k(\bar t_k)] \ge \prod_{i\in J\setminus J_{2s}} g_i(\bar t_i) \prod_{k\in M\setminus (J\setminus J_{2s})}[g_i(d_i)-g_i(\bar t_i)],$$

that after cancellation of the terms will reduce to (17).

For proof of (18) it should take

$$J=I\cup I^*_{2r}, \qquad Ж=J\cup J^*_{2s},$$

where $I^*_{2r}\subset M\setminus I$, $J^*_{2s}\subset M\setminus J$ and act as in the case of proof of (17).

Lemma 2 has been proved.

Corollary. *For an arbitrary couple of indices (i,k) belonging to the sets I and J it is valid*

$$g_i(\bar t_i)g_k(\bar t_k) \ge [g_i(d_i)-g_i(\bar t_i)][g_k(d_k)-g_k(\bar t_k)], \tag{19}$$

and for any couple (i,k) belonging $M\setminus I$ or $M\setminus J$ it is valid

$$[g_i(d_i)-g_i(\bar t_i)]\,[g_k(d_k)-g_k(\bar t_k)] \ge g_i(\bar t_i)g_k(\bar t_k). \tag{20}$$

Lemma 3. *For $i\in I\setminus J$ and $k\in J\setminus I$ it takes place the equality*

$$g_i(\bar t_i)[g_k(d_k)-g_k(\bar t_k)] = [g_i(d_i)-g_i(\bar t_i)]g_k(\bar t_k). \tag{21}$$

Proof. Let $i_0\in I$ and $k_0\in M\setminus I$. Supposing $\mathcal{J}^*=(I\setminus i_0)\cup K_0$ we write (15)

$$\prod_{i\in I} g_i(\bar t_i) \prod_{k\in M\setminus I}[g_k(d_k)-g_k(\bar t_k)] \ge \prod_{i\in \mathcal{J}^*} g_i(\bar t_i) \prod_{k\in M\setminus \mathcal{J}^*}[g_k(d_k)-g_k(\bar t_k)]$$

whence after cancellations we will obtain

$$g_{i_0}(\bar t_{i_0})[g_k(d_{k_0})-g_{k_0}(\bar t_{k_0})] \ge [g_{i_0}(d_{i_0})-g_{i_0}(\bar t_{i_0})]g_{k_0}(\bar t_{k_0}). \tag{22}$$

It is clear, that in (22) instead i_0, k_0 we can put arbitrary $i_0\in I$, $k_0\in M\setminus I$. Making analogous operations with (16) we will obtain for arbitrary

$$g_i(\bar{t}_i)[g_k(d_k)-g_k(\bar{t}_k)]\geq[g_i(d_i)-g_i(\bar{t}_i)]g_k(\bar{t}_k)\,. \quad (23)$$

Not assume $i\in I\setminus J$, $k\in J\setminus I$. As far as $i\in I\setminus J\Rightarrow i\in I$; $k\in J\setminus I\Rightarrow k\notin I\Rightarrow$ $\Rightarrow k\in M\setminus I$ then for these i and k (22) is valid. As far as $i\in I\setminus J\Rightarrow i\in M\setminus J$; $k\in J\setminus I\Rightarrow k\in J$, then in (23) i has a role of k and k has a role of i. So from (23) we obtain for $i\in I\setminus J$, $k\in J\setminus I$

$$g_k(\bar{t}_k)[g_i(d_i)-g_i(\bar{t}_i)]\geq[g_k(d_k)-g_k(\bar{t}_k)]g_i(d_i). \quad (24)$$

Comparing (22) with (24) we obtain for (21) Lemma 3 has been proved.

Lemma 4. *Let $I\setminus J\neq\varnothing$, $mes(J\setminus I)>1$ or $J\setminus I\neq\varnothing$, $mes(I\setminus J)>1$. Then*

$$g_i(\bar{t}_i)\geq g_i(d_i)-g_i(\bar{t}_i),\quad i\in(I\setminus J)\cup(J\setminus I). \quad (25)$$

Proof. Let $I\setminus J\neq\varnothing$, $mes(J\setminus I)>1$. Assume for some $i_0\in I\setminus J$ in spite of (25) it takes place the inequality

$$g_{i_0}(\bar{t}_{i_0})<g_{i_0}(d_{i_0})-g_{i_0}(\bar{t}_{i_0}). \quad (26)$$

Write (21) for $i=i_0$ and all $k\in J\setminus I$

$$g_{i_0}(\bar{t}_{i_0})[g_k(d_k)-g_k(\bar{t}_k)]=[g_{i_0}(d_{i_0})-g_{i_0}(\bar{t}_{i_0})]g_k(\bar{t}_k).$$

Then by (26) we will have for all $k\in J\setminus I$

$$g_k(d_k)-g_k(\bar{t}_k)>g_k(\bar{t}_k)$$

and then for the couple $i,k\in J\setminus I$ we obtain

$$[g_i(d_i)-g_i(\bar{t}_i)]\,[g_k(d_k)-g_k(\bar{t}_k)]>g_i(\bar{t}_i)g_k(\bar{t}_k)$$

that contradicts to (19) from consequence of lemma 2. The second case is considered by analogy. Lemma 4 has been proved.

Let's go on with proof of the theorem. The equalities (13) and (14) let write

$$\prod_{i\in I}g_i(\bar{t}_i)\prod_{k\in M\setminus I}[g_k(d_k)-g_k(\bar{t}_k)]=\prod_{i\in J}g_i(\bar{t}_i)\prod_{k\in M\setminus J}[g_k(d_k)-g_k(\bar{t}_k)]. \quad (27)$$

In (27) there are the same multipliers $g_i(\bar{t}_i)$ for $i\in I\cap J$ and $g_k(d_k)-g_k(\bar{t}_k)$ for $k\in(M\setminus I)\cap(M\setminus J)$.

After cancellations these multiplies in (27) the equality is obtained

$$\prod_{i\in I\setminus(I\cap J)}g_i(\bar{t}_i)\prod_{k\in(M\setminus I)\setminus[(M\setminus I)\cap(M\setminus J)]}[g_k(d_k)-g_k(\bar{t}_k)]=$$
$$=\prod_{i\in J\setminus(I\cap J)}g_i(\bar{t}_i)\prod_{k\in(M\setminus J)\setminus[(M\setminus I)\cap(M\setminus J)]}[g_k(d_k)-g_k(\bar{t}_k)],$$

which by Lemma 1, a and 1, b can be written in the form

$$\prod_{i\in I\setminus J}g_i(\bar{t}_i)\prod_{k\in J\setminus I}[g_k(d_k)-g_k(\bar{t}_k)]=\prod_{i\in J\setminus I}g_i(\bar{t}_i)\prod_{k\in I\setminus J}[g_k(d_k)-g_k(\bar{t}_k)]$$

or redenoting i and k:

$$\prod_{i\in I\setminus J}g_i(\bar{t}_i)\prod_{k\in J\setminus I}[g_k(d_k)-g_k(\bar{t}_k)]=\prod_{i\in I\setminus J}[g_i(d_i)-g_i(\bar{t}_i)]\prod_{k\in J\setminus I}g_k(\bar{t}_k). \quad (28)$$

Let $mesI\setminus J=p$, $mesJ\setminus I=q$. For determinity let assume $p>q$ (by virtue of lemma 1, $p\neq q$)

$$I\setminus J=\{i_1,\dots,i_p\},\quad J\setminus I=\{i_{p+1},\dots,i_{p+q}\}\quad .$$

Let us consider the product q of the equalities of the form (21):

$$\prod_{i=i_1,\dots,i_q}g_i(\bar{t}_i)\prod_{k=i_{p+1},\dots,i_{p+q}}[g_k(d_k)-g_k(\bar{t}_k)]=\prod_{i=i_1,\dots,i_q}[g_i(d_i)-g_i(\bar{t}_i)]\prod_{k=i_{p+1},\dots,i_{p+q}}g_k(\bar{t}_k). \quad (29)$$

Canceling (28) for (29) we obtain

$$\prod_{i=i_{q+1},\dots,i_p}g_i(\bar{t}_i)=\prod_{i=i_{q+1},\dots,i_p}[g_i(d_i)-g_i(\bar{t}_i)]. \quad (30)$$

As far as $\{i_{q+1},...,i_p\} \subset I \setminus J$, then by lemma 4

$$g_i(t_i) > g_i(d_i) - g_i(\bar{t}_i), \quad i = i_{q+1},...,i_{q+p}.$$

Let for some $i_* \in \{i_{q+1},...,i_p\}$ the strong inequality has place:

$$g_{i_*}(\bar{t}_{i_*}) > g_{i_*}(d_{i_*}) - g_{i_*}(\bar{t}_{i_*}).$$

Then for saving of (30) it must exist

$$i_{**} \in \{i_{q+1},...,i_{q+p}\}$$

for which

$$g_{i_{**}}(\bar{t}_{i_{**}}) < g_{i_{**}}(d_{i_{**}}) - g_{i_{**}}(\bar{t}_{i_{**}}),$$

that contradicts lemma 4. Consequently, for arbitrary $i \in \{i_{q+1},...,i_p\}$

$$g_i(\bar{t}_i) = g_i(d_i) - g_i(\bar{t}_i) \tag{31}$$

or

$$g_i(\bar{t}_i) = \frac{1}{2} g_i(d_i). \tag{32}$$

It is clear that instead of $\{i_1,...,i_q\}$ selected other subset from $\{i_1,...,i_p\}$ consisting of q elements and repeating the above-given procedure we will obtain that the equalities (31) and (32) are valid for all $i \in \{i_1,...,i_p\}$ or $i \in I \setminus J$.

Now by (31) we obtain

$$\prod_{i\in I\setminus J} g_i(\bar{t}_i) = \prod_{i\in I\setminus J} [g_i(d_i) - g_i(\bar{t}_i)]. \tag{33}$$

Canceling (28) for (33) we obtain

$$\prod_{k\in J\setminus I} [g_k(d_k) - g_k(\bar{t}_k)] = \prod_{k\in J\setminus I} g_k(\bar{t}_k)$$

Hence, as above, by Lemma 4 it is obtained

$$g_k(\bar{t}_k) = g_k(d_k) - g_k(\bar{t}_k), \quad k \in J \setminus I$$

or

$$g_k(t_k) = \frac{1}{2} g_k(d_k), \quad k \in J \setminus I.$$

Combining the last two equalities with (31) and (32) we obtain, that the next lemma is valid.

Main lemma. *For $I \setminus J \neq \varnothing$, $mes(J \setminus I) > 1$ or $J \setminus I \neq \varnothing$, $mes(I \setminus J) > 1$ the equalities are valid*

$$g_k(\bar{t}_k) = \frac{1}{2} g_k(d_k), \quad k \in (I \setminus J) \cup (J \setminus I). \tag{34}$$

In order to obtain the equalities of the form (34) also for other $i \in M$ let us use the correlation

$$\prod_{i\in I} g_i(\bar{t}_i) \prod_{k\in M\setminus I} [g_k(d_k) - g_k(\bar{t}_k)] = 2^{-m} \prod_{1}^{m} g_v(d_v). \tag{35}$$

Write (33) in the form

$$g_k(d_k) - g_k(\bar{t}_k) = \frac{1}{2} g_k(d_k), \quad k \in (I \setminus J) \cup (J \setminus I). \tag{36}$$

Canceling the equality (35) to (34) for $k \in I \setminus J$ (in this case in (34) index k was substituted by index i) and to (36) for $k \in J \setminus I$ we obtain

$$\prod_{i\in I\setminus(I\setminus J)} g_i(\bar{t}_i) \prod_{k\in(M\setminus I)\setminus(J\setminus I)} [g_k(d_k) - g_k(\bar{t}_k)] = 2^{-\{m - mes[(I\setminus J)\cup(J\setminus I)]\}} \prod_{s\in M\setminus\{(I\setminus J)\cup(J\setminus I)\}} g_s(d_s). \tag{37}$$

Using Lemma 1,d and 1,e the equality (37) can be written as

$$\prod_{i\in I\cap J} g_i(\bar{t}_i) \prod_{k\in(M\setminus I)\cap(M\setminus J)}[g_k(d_k)-g_k(\bar{t}_k)] = 2^{-\{m-mes[(I\setminus J)\cup(J\setminus I)]\}} \prod_{s\in M\setminus[(I\setminus J)\cup(J\setminus I)]} g_s(d_s). \quad (38)$$

Let for the conditions of the Main lemma $i\in I\setminus J$, $r\in I\cap J$. By virtue of the Main lemma

$$g_i(\bar{t}_i)=g_i(d_i)-g_i(\bar{d}_i), \quad (34')$$

and by the corollary of Lemma 2 (as far as $i,r\in I$)

$$g_i(\bar{t}_i)g_r(\bar{t}_r)\geq [g_i(d_i)-g_i(\bar{d}_i)]\,[g_r(d_r)-g_r(\bar{t}_r)].$$

Cancelling the last inequality for (34) we obtain

$$g_r(\bar{t}_r)\geq g_r(d_r)-g_r(\bar{t}_r), \quad \forall r\in I\cap J. \quad (39)$$

Let $i\in I\setminus J$ (it is clear that one of the sets $I\setminus J$ and $J\setminus I$ is not empty for determinancy let $I\setminus J\neq\varnothing$), let us take $k\in(M\setminus J)\cap(M\setminus J)$. By the main lemma the equality (34) is valid and as far as $i\in I\setminus J\Rightarrow k\in M\setminus J$, by the corollary of Lemma 2 it is valid

$$[g_i(d_i)-g_i(\bar{t}_i)]\,[g_k(d_k)-g_k(\bar{t}_k)]\geq g_i(\bar{t}_i)d_k(\bar{t}_k).$$

Cancelling the last inequality for (34) we obtain

$$g_k(d_k)-g_k(\bar{t}_k)\geq g_k(\bar{t}_k) \quad (40)$$

for all $k\in(M\setminus I)\cap(M\setminus J)$.

Let's assume that for some $i_0\in I\cap J$ the strong inequality has place:

$$g_{i_0}(\bar{t}_{i_0})>\frac{1}{2}g_{i_0}(d_{i_0}). \quad (41)$$

Then for saving of (38) it must exist the index $i_*\in I\cap J$, which for

$$g_{i_*}(\bar{t}_{i_*})<\frac{1}{2}g_{i_*}(d_{i_*}) \quad (42)$$

or it must exist the index $k_*\in(M\setminus I)\cap(M\setminus J)$, which for

$$g_{k_*}(d_{k_*})-g_{k_*}(\bar{t}_{k_*})<\frac{1}{2}g_{k_*}(d_{k_*}). \quad (43)$$

However, (42) contradicts (39) and (43) contradicts (40). Consequently, it must not be the strong inequality (39) and we have

$$g_i(\bar{t}_i)=\frac{1}{2}g_i(d_i) \quad (44)$$

for all $i\in I\cap J$.

By analogy, it there exists the index $k\in(M\setminus I)\cap(M\setminus J)$, which for

$$g_k(d_k)-g_k(\bar{t}_k)>\frac{1}{2}g_k(\bar{t}_k),$$

then it must exist $i_*\in I\cap J$, satisfying (42) or $k_*\in(M\setminus I)\cap(M\setminus J)$ satisfying (43), which as it was above mentioned can not have place. Thus, for all $k\in(M\setminus I)\cap(M\setminus J)$ it is valid (44). By Lemma 1,e the Theorem has been proved for all $i\in M$ for the conditions of Lemma 4. Consideration of the rest of the cases has a technical character.

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