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MATHEMATICS

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ON THE NORMS AND LEADING EIGEN-VALUES OF NON-NEGATIVE MATRICES OF LARGE DIMENSION

Abstract

The set of nonnegative matrices of the large dimension is investigated and it proves that for the large dimensions the norm and main eigen-value of almost all matrices are in the small neighbourhood of some "point".

It is known that large dimension systems possess typical properties. It these systems by increasing the dimension, some parameters are stabilized. For instance, in physical; systems, where the dimension in calculated by the Avogadro number degree, the relations between the macro parameters (such as pressure, temperature and volume), are known and they exactly reflect the system state.

In this aspect, it is important to study a class of non-negative matrices of large dimension that find their essential application in mathematical economic, probability theory, small oscillations theory of elastic systems and etc. In the paper, it is shown that for a definite set from this class there is such a positive member in whose small neighborhood, leading eigen-values and norms of majority of matrices from this set are concentrated. In addition, by increasing the dimension, the conden the densability around "the concentration point" grows. Note that "the concentration point" depends on geometry of the studied set of real matrices of order n, considered as a subset of n^2-dimensional Euclidean space.

Let $M(n)$ be a set of square non-negative matrices of order n,

$$K(n;h)=\left\{A\in M(n)\middle| 0\le a_{ij}\le h,\quad i,j=\overline{1,n}\right\},$$

$$G(n;h,\varepsilon,\mu)=\left\{A\in K(n;h)\middle\|\left|\frac{\lambda(A)}{n}-\mu\right|\le\varepsilon\right\},$$

$$\Phi(n;h,\varepsilon,\mu)=\left\{A\in K(n;h)\middle\|\left|\frac{\|A\|}{n}-\mu\right|\le\varepsilon\right\},$$

where $\lambda(A)$ is the leading eigen-value of the matrix A; μ,ε,h are the given numbers satisfying the following conditions: $\mu>\varepsilon>0$, $\mu+\varepsilon\le h$.

Theorem. *Let be given as much as desired small* $\varepsilon>0$. *Then*

$$\left[1-\frac{h^4}{\varepsilon^4}\left(\frac{1}{48n^2}-\frac{1}{120n^3}\right)\right]^n\theta\left(1-\frac{h^4}{\varepsilon^4}\left(\frac{1}{48n^2}-\frac{1}{120n^3}\right)\right)\le\frac{mes\Phi(n;h,\varepsilon,\mu)}{mesK(n;h)}\le$$
$$\le\frac{mesG(n;h,\varepsilon,\mu)}{mesK(n;h)}\le 1 \qquad (1)$$

for $\mu=\frac{h}{2}$, *where* $\theta(x)$ *is a Heavyside function, i.e.* $\theta(x)=\begin{cases}1 & if \quad x>0\\ 0 & if \quad x\le 0\end{cases}$.

Proof. The following inequalities hold for non-negative matrices:

$$\min_i \sum_{j=1}^{n} a_{ij} \le \lambda(A) \le \max_i \sum_{j=1}^{n} a_{ij} = \|A\|.$$

Let

$$\mathcal{F}(n;h,\varepsilon,\mu)=\left\{A\in K(n;h)\left|\left|\frac{\sum_j a_{ij}}{n}-\mu\right|\le\varepsilon,\quad i=\overline{1,n}\right.\right\}.$$

Then it is clear that

$$\mathcal{F}(n;h,\varepsilon,\mu)\subset\Phi(n;h,\varepsilon,\mu)\subset G(n;h,\varepsilon,\mu)\subset K(n;h)$$

and it is sufficient to show that

$$\left[1-\frac{h^4}{\varepsilon^4}\left(\frac{1}{48n^2}-\frac{1}{120n^3}\right)\right]^n\theta\left(1-\frac{h^4}{\varepsilon^4}\left(\frac{1}{48n^2}-\frac{1}{120n^3}\right)\right)\le\frac{mes\mathcal{F}(n;h,\varepsilon,\mu)}{mesK(n;h)}\le 1 \qquad (2)$$

for $\mu=\dfrac{h}{2}$.

Calculate $mes\mathcal{F}(n;h,\varepsilon,\mu)$. Since $a_{ij}\ (i,j=\overline{1,n})$ are independent variables, we get

$$mes\mathcal{F}(n;h,\varepsilon,\mu)=\overbrace{\int\cdots\int}^{n^2}_{\substack{\left|\frac{\sum_j a_{ij}}{n}-\mu\right|\le\varepsilon,\ i=\overline{1,n}\\ 0\le a_{ij}\le h,\ i,j=\overline{1,n}}} da_{11}da_{12}\ldots da_{nn}=\left[\overbrace{\int\cdots\int}^{n}_{\substack{\left|\frac{\sum_j a_{1j}}{n}-\mu\right|\le\varepsilon\\ 0\le a_{1j}\le h,\ j=\overline{1,n}}} da_{11}da_{12}\ldots da_{1n}\right]^n=$$

$$=\left[h^n-\overbrace{\int\cdots\int}^{n}_{\substack{\left|\frac{\sum_j a_{1j}}{n}-\mu\right|>\varepsilon\\ 0\le a_{1j}\le h,\ j=\overline{1,n}}} da_{11}da_{12}\ldots da_{1n}\right]^n.$$

Upper estimate for $V(n;h,\varepsilon,\mu)=\overbrace{\int\cdots\int}^{n}_{\substack{\left|\frac{\sum_j a_{1j}}{n}-\mu\right|>\varepsilon\\ 0\le a_{1j}\le h,\ j=\overline{1,n}}} da_{11}da_{12}\ldots da_{1n}$ gives an integral taken on the domain

$$\left(\frac{\sum_j a_{1j}}{n}-\mu\right)^4>\varepsilon^4,\quad 0\le a_{1j}\le h,\quad j=\overline{1,n}, \qquad (3)$$

i.e.

$$V(n;h,\varepsilon,\mu)\le \overbrace{\int\limits_{\substack{\left(\frac{\sum_j a_{1j}}{n}-\mu\right)^4>\varepsilon^4\\ 0\le a_{1j}\le h,\ j=\overline{1,n}}}\cdots\int}^{n} da_{11}da_{12}\dots da_{1n}.$$

It is valid also the following equality:

$$\int_0^h\cdots\int_0^h\left(\frac{\sum_j a_{1j}}{n}-\mu\right)^4 da_{11}da_{12}\dots da_{1n}\ge\varepsilon^4\int\limits_{\substack{\left(\frac{\sum_j a_{1j}}{n}-\mu\right)^4>\varepsilon^4\\ 0\le a_{1j}\le h,\ j=\overline{1,n}}}\cdots\int d_{11}da_{12}\dots da_{1n}.$$

Hence it follows that

$$V(n;h,\varepsilon,\mu)\le\frac{1}{\varepsilon^4}\int_0^h\cdots\int_0^h\left(\frac{\sum_j a_{1j}}{n}-\mu\right)^4 da_{11}da_{12}\dots da_{1n}.$$

Now calculate the integral at the right-hand side

$$\int_0^h\cdots\int_0^h\left(\frac{\sum_j a_{1j}}{n}-\mu\right)^4 da_{11}da_{12}\dots da_{1n}=$$

$$=\frac{1}{n_4}\int_0^h\cdots\int_0^h\left[\left(\sum_j a_{1j}\right)^4-4\mu n\left(\sum_j a_{1j}\right)^3+6\mu^2n^2\left(\sum_j a_{1j}\right)^2-4\mu^3n^3\sum_j a_{1j}+\mu^4n^4\right]\times \tag{4}$$

$$\times\, da_{11}da_{12}\dots da_{1n}$$

$$\int_0^h\cdots\int_0^h\left(\sum_j a_{1j}\right)^4 da_{11}da_{12}\dots da_{1n}=\int_0^h\cdots\int_0^h\sum_{\substack{i\ne j\ne k\ne l,\\ i\ne k,\ i\ne l,\\ j\ne l}}a_{1j}a_{1i}a_{1k}a_{1l}+6\sum_{\substack{i\ne j\ne k,\\ i\ne k,}}a_{1j}^2a_{1i}a_{1k}+$$

$$\left.+3\sum_{i\ne j}a_{1j}^2a_{1j}^2+4\sum_{i\ne j}a_{1j}^3a_{1j}+\sum_j a_{1j}^4\right)da_{11}da_{12}\dots da_{1n}=n(n-1)(n-2)(n-3)\frac{h^{n+4}}{16}+ \tag{5}$$

$$+6n(n-1)(n-2)\frac{h^{n+4}}{12}+3n(n-1)\frac{h^{n+4}}{9}+4n(n-1)\frac{h^{n+4}}{8}+n\frac{h^{n+4}}{5}=$$

$$=h^{n+4}\left(\frac{n(n-1)(n-2)(n-3)}{16}+\frac{n(n-1)(n-2)}{2}+\frac{5n(n-1)}{6}+\frac{n}{5}\right).$$

Analogously

$$\int_0^h\cdots\int_0^h\left(\sum_j a_{1j}\right)^3 da_{11}da_{12}\dots da_{1n}=h^{n+3}\left(\frac{n(n-1)(n-2)}{8}+\frac{n(n-1)}{2}+\frac{n}{4}\right), \tag{6}$$

$$\int_0^h\cdots\int_0^h\left(\sum_j a_{1j}\right)^2 da_{11}da_{12}\dots da_{1n}=h^{n+2}\left(\frac{n(n-1)}{4}+\frac{n}{3}\right), \tag{7}$$

$$\int_0^h\cdots\int_0^h\sum_j a_{1j}da_{11}da_{12}\dots da_{1n}=h^{n+1}\frac{n}{2} \tag{8}$$

are calculated.

Thus, it follows from (4), (5), (6), (7), (8) that

$$\int_0^h \cdots \int_0^h \left(\frac{\sum_j a_{1j}}{n} - \mu \right)^4 da_{11} da_{12} \ldots da_{1n} = h^4 \left[\left(\mu - \frac{h}{2} \right)^4 + \left(\mu - \frac{h}{2} \right)^2 \frac{h^2}{2n} + \frac{h^4}{48n^2} - \frac{h^4}{120n^3} \right].$$

Thus, we get

$$V(n;h,\varepsilon,\mu) \le \frac{h^4}{\varepsilon^4} \left[\left(\mu - \frac{h}{2} \right)^4 + \left(\mu - \frac{h}{2} \right)^2 \frac{h^2}{2n} + \frac{h^4}{48n^2} - \frac{h^4}{120n^3} \right].$$

Then, it is valid the following inequality

$$mes\mathcal{F}(n;h,\varepsilon,\mu) \ge h^{n^2} \left\{ 1 - \frac{1}{\varepsilon^4} \left[\left(\mu - \frac{h}{2} \right)^4 + \left(\mu - \frac{h}{2} \right)^2 \frac{h^2}{2n} + \frac{h^4}{48n^2} - \frac{h^4}{120n^3} \right] \right\}^n \times$$
$$\times \theta \left[1 - \frac{1}{\varepsilon^4} \left(\left(\mu - \frac{h}{2} \right)^4 + \left(\mu - \frac{h}{2} \right)^2 \frac{h^2}{2n} + \frac{h^4}{48n^2} - \frac{h^4}{120n^3} \right) \right]. \quad (9)$$

The expression at the right hand side achieves its maximum (with respect to the parameter μ) for $\mu = \frac{h}{2}$. Thus, (2) follows from (9). The Theorem is proved.

It follows from Theorem that

$$\lim_{n\to\infty} \frac{mesG(n;h,\varepsilon,\mu)}{mesK(n;h)} = \lim_{n\to\infty} \frac{mes\Phi(n;h,\varepsilon,\mu)}{mesK(n;h)} = 1$$

for $\mu = \frac{h}{2}$.

Remark 1. Note that, if for the estimation $V(n;h,\varepsilon,\mu)$ instead of (3) we take the domain

$$\left(\frac{\sum_j a_{1j}}{n} - \mu \right)^2 > \varepsilon^2, \quad 0 \le a_{1j} \le h, \quad j = \overline{1,n},$$

the estimation will be rough and we can't obtain the desired result, as

$$\int_0^h \cdots \int_0^h \left(\frac{\sum_j a_{1j}}{n} - \mu \right)^2 da_{11} da_{12} \ldots da_{1n} = h^n \left[\left(\mu - \frac{h}{2} \right)^2 + \frac{h^2}{12n} \right]$$

and then

$$\left\{ 1 - \frac{1}{\varepsilon^2} \left[\left(\mu - \frac{h}{2} \right)^2 + \frac{h^2}{12n} \right] \right\}^n \theta \left[1 - \frac{1}{\varepsilon^2} \left(\left(\mu - \frac{h}{2} \right)^2 + \frac{h^2}{12n} \right) \right] \le$$
$$\le \frac{mes\Phi(n;h,\varepsilon,\mu)}{mesK(n;h)} \le \frac{mesG(n;h,\varepsilon,\mu)}{mesK(n;h)} \le 1.$$

For $\mu = \frac{h}{2}$, at the left side will be the expression $\left(1 - \frac{h^2}{12n\varepsilon^2} \right)^n \theta \left(1 - \frac{h^2}{12n\varepsilon^2} \right)$, which converges to $e^{-\frac{h^2}{12\varepsilon^2}}$ for $n \to \infty$.

Remark 2. We remarked above that "the concentration point" depends on geometry of the given set of matrices. Consider the following set

$$\Omega(n;h)=\{A\in M(n)\|\|A\|\le h\}\subset K(n;h).$$

Let

$$\widetilde{G}(n;h,\varepsilon,\mu)=\{A\in\Omega(n;h)\,|\,|\lambda(A)-\mu|\le\varepsilon\},$$

$$\widetilde{\Phi}(n;h,\varepsilon,\mu)=\{A\in\Omega(n;h)\,|\,|\|A\|-\mu|\le\varepsilon\},$$

where ε,μ are given members satisfying the conditions $\mu>\varepsilon>0$ and $\mu+\varepsilon\le h$.

Similarly we can show that

$$\left(\frac{\mu+\varepsilon}{h}\right)^{n^2}\left[1-\left(\frac{\mu-\varepsilon}{\mu+\varepsilon}\right)^{n}\right]^{n}\le\frac{mes\widetilde{\Phi}(n;h,\varepsilon,\mu)}{mes\Omega(n;h)}\le\frac{mes\widetilde{G}(n;h,\varepsilon,\mu)}{mes\Omega(n;h)}\le 1.$$

It is clear that the expression at the left-hand side converges to the unit for $n\to\infty$, when $\mu+\varepsilon=h$, and in other cases it converges to zero. So, "the concentration point" here will be $\mu=h$, moreover "the concentration" happens at the segment $[h-\varepsilon,h]$.

Note that $\Omega(n;1)$ (i.e. for $h=1$) represents a set of matrices used in description of interbranch balance model and was investigated within such approach in paper [1].

Remark 3. We may show that by rejecting from the non-negativity of matrices, the obtained result on the norms of matrices is valid and in the set $\widetilde{K}(n;h)=\{A\,|\,|a_{ij}|\le h,\ i,j=\overline{1,n}$, i.e. and in other orthants of the n^2-dimensional Euclidean space.

References

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