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BAIMOV Sh.K.

THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF NONSELF-ADJOINT DIFFERENTIAL OPERATORS IN UNBOUNDED DOMAINS WITH BOUNDARY

Abstract

Boundary value problems are considered in an unbounded domain for nonself-adjoint operators of elliptic types. The Green's function are constructed and the asymptotic formula for a number of eigen-values is obtained, where all coefficients are used.

Spectral properties of elliptic operators have been studied by many researchers (see survey works [1]-[6]). Asymptotic formulas characterizing the distribution of eigen-values of self-adjoint elliptic operators defined on the whole space have been considered in papers [4]-[7]. The matters on distribution of spectrum for the operators generated by general elliptic boundary value problems have been stated in monographs [4]-[7] and in [8]. Non self- adjoint elliptic higher order problems have been studied in papers [9], [10]. In [9], asymptotic behavior of Green's function up to the boundary of unbounded domain has been studied, asymptotic formulas have been obtained for the distribution of eigen-values, where all leading and free coefficients of non-self-adjoint elliptic equation participate. In paper [10] asymptotic formulas where intermediate coefficients also take part under rigid conditions on their growth, were proved by means of bilinear forms theory method. It is assumed that the boundary of the domain satisfies the condition of cone [6].

Asymptotic distribution of eigen-values of a non- self-adjoint operator generated by a general elliptic boundary value problem in an unbounded domain is investigated in the paper.

Here the author's results [9] are generalized, namely, some conditions on the coefficients of a differential expression have been weakened; intermediate terms may have definite order discontinuity, besides, new characteristics for the asymptotics of spectrum were suggested. The properties of Green's functions of the considered boundary value problem, and also Green's functions of corresponding parabolic boundary value problem were studied. By means of this method one can determine and asymptotics of a spectral function, asymptotics of eigen functions. And finally, uniform asymptotics of Green's functions admits to determine the form of domain where a discrete spectrum of the studied operator lies.

Let $\Omega \in R^n$ be an arbitrary (bounded or unbounded) domain with the boundary $\Gamma \in C^m$ [4].

The differential operator L given by the inequality

$$Lu = \sum_{|\alpha| \le b} a_\alpha(x) D^\alpha u \qquad (1.1)$$

is called strongly elliptic on Ω, if for all $x \in \Omega$ and for any real vector $0 \ne \xi \in R^n$

$$\operatorname{Re}\left\{ \sum_{|\alpha| = b} a_\alpha(x) \xi^\alpha \right\} \ge c_0 \|\xi\|^b , \quad c_0 > 0 . \qquad (1.2)$$

Here

$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}$, $D_k = \frac{1}{i}\frac{\partial}{\partial x^k}$; $(k = \overline{1,n})$, $i = \sqrt{-1}$, $\alpha = (\alpha_1; \alpha_2; \dots; \alpha_n)$ is a multi-index and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$.

We are to note that for $n \ge 3$ the order of the operator b is even. We shall assume that the form of leading terms

$$L_0(x,\xi) = \sum_{|\alpha|=b} a_\alpha(x)\xi^\alpha$$

is uniformly elliptic, i.e., for all $x \in \Omega$ and real $\xi = (\xi_1; \xi_2; \dots; \xi_n) \ne 0$

$$m_1\|\xi\|^b \le \left\| \sum_{|\alpha|=b} a_\alpha(x)\xi_1^{\alpha_1}\xi_2^{\alpha_2}\dots\xi_n^{\alpha_n} \right\| \le m_2\|\xi\|^b, \tag{1.3}$$

for some $m_1 > 0$, $m_2 > 0$, that do not depend on $x \in \Omega$.

1. In a class of m- dimensional vector-functions it is considered a boundary value problem on eigen values:

$$Lu \equiv L_0 u + L_1 u + qu =$$

$$= (-1)^b \sum_{\substack{|k|=b \\ |j|=b}} D^j(a_k(x)D^k u) + \sum_{1 \le k \le 2b-1} b_k(x)D^k u + q(x)u = \lambda\, r(x)u\ , \tag{1.4}$$

$$\Phi_s(x,D)u|_\Gamma = 0, \quad s = \overline{1,b}, \tag{1.5}$$

where the expression $\Phi_s(x,D)$ satisfy the Ya.B. Lopatinsky conditions [4], in other words, if

$$\Phi_j u = \sum_{|\alpha| \le m_j} b_{j,\alpha}(x)D^\alpha u,$$

$$b_{j,\alpha}(x) \in C^\infty(\Gamma), \quad j = \overline{1,k}$$

for any vector ν_x, normal to the boundary Γ at the point $x \in \Gamma$ the conditions

$$\sum_{|\alpha|=m_j} b_{j,\alpha}(x)\nu_x^\alpha \ne 0;\ j = \overline{1,k},$$

$$0 \le m_1 < m_2 < \dots < m_k \le 2b-1;$$

are fulfilled and this set forms an additional system with respect to L [6].

The differential expression

$$L_1(x,D) = \sum_{1 \le k \le 2b-1} b_k(x)D^k$$

has, in general complex valued coefficients, they may have definite order discontinuities:

$$|b_k(x)| \le m_3 Q_k(x), \tag{1.6}$$

where

$$Q_k(x) = \begin{cases} |x|^{-k};\ \text{for } k > 0;\ \|x\| < 1 \\ -\ln|x|;\ \text{for } k = 0;\ \|x\| < 1 \\ 1;\ \text{for } k > 0;\ \|x\| \ge 1 \end{cases}$$

It is assumed that the square matrix $r(x) \in C_2(\Omega)$ is positive definite. Let $\lambda_j(x) = l_j(x) + it_j(tx)$ be characteristical roots of the matrix $r^{-1}(x)q(x)$, and elements $l_j(x) > 0$, $j = \overline{1,n}$ are arranged at decreasing order. We shall assume that

$$\|L_0(x,\xi) - L_0(y,\xi)\| \le m_4\|x-y\|^\gamma \|\xi\|^{2b};\ \|x-y\| \le 1,\ \ 0 < \gamma < 1, \tag{1.7}$$

$$|q(x)-q(y)| \le m_5 r(x) l_1^{\alpha}(y) \|x-y\|, \tag{1.8}$$

$$|b_k(x)| \le m_6 [l_1(x)]^{\frac{2b-|k|}{2b}-\varepsilon} \quad \text{for} \quad \|x\| \to \infty; \tag{1.9}$$

where $\alpha = 1 + \frac{1}{2b}$;

there exists such a number $p > 0$, that

$$\int_{\Omega} l_1^{-p}(x) dx < \infty,$$

$$l_n(x) \le [l_1(x)]^{\beta}; \ \|x-y\| > 1, \ \beta \le 1. \tag{1.10}$$

Seeley [11], and further Mizohata suggested a method to study spectral properties of the problem (1.4)-(1.5) by means of studying Green's functions properties of the parabolic problem

$$\frac{\partial u}{\partial t} + Lu = 0, \quad \text{into } \Omega_1 = \Omega \oplus [0;\tau], \tag{1.11}$$

$$u|_{t=0} = f(x), \tag{1.12}$$

$$\Phi_{\nu}(x, D_x) u|_{\Gamma_1} = 0, \quad \Gamma_1 = \Gamma \oplus [0,\tau]. \tag{1.13}$$

Let $G_0(x-y,y,t)$ be a fundamental matrix for the solution of the system (1.11) for $Lu = L_0 u + q(y)u$. Then

$$G_0(x-y,y,t) = \frac{1}{(2\pi)^n t^{\frac{n}{2b}}} \int_{R^n} \left\{ \exp\left[-\sum_{|k|=2b} a_{k_1 k_2 \ldots k_n}(y) \xi_{k_1}^{k_1} \cdot \xi_{k_2}^{k_2} \cdot \ldots \cdot \xi_{k_n}^{k_n} + \right.\right.$$
$$\left.\left. + i \sum_{|k|=1}^{n} \xi_k \frac{|x_k - y_k|}{t^{\frac{1}{2b}}} - \lambda_k(y)\, t \right\} d\xi \tag{1.14}$$

and the Green's function of the problem (1.11)-(1.13) is represented as (see [9])

$$G(x,y,t) = G_0(x-y,y,t) + G_1(x,y,t) + G_2(x,y,t), \tag{1.15}$$

where

$$\|G_1(x,y,t)\| \le m_7 t^{-\frac{n}{2b}} \exp\left\{ -m_8 \frac{\|x-y\|^q}{t^{\frac{1}{2b-1}}} - t l_j(y) \right\}, \tag{1.16}$$

$$\|G_2(x,y,t)\| \le m_9 t^{-\frac{n}{2b}} \exp\left\{ -m_{10} [\|x-y\| - \rho(y,\Gamma)]^q t^{-\frac{1}{2b-1}} - t l_j(y) \right\}, \tag{1.17}$$

where $\frac{1}{2b} + \frac{1}{q} = 1$, $m_i > 0$; $\rho(y,\Gamma)$ is the distance from the point $y \in \Omega$ to the boundary Γ. Inequalities (1.15)-(1.17) are proved by the schemes of papers [8] and [12].

Theorem 1. *Non-self-adjoint operator L, corresponding to the problem (1.4)-(1.5) and satisfying the abovestated conditions, possesses a full system of eigen and adjoint functions.*

Theorem 2. *If the conditions of Theorem 1 are fulfilled, and besides*

$$\left\| (L_0 + l_j t_j u, u)^{-1} \right\| \le m_{11} \left\| (L_0 + l_j)^{-\delta} u \right\|, \quad j = \overline{1,m},$$

the spectrum of the operator L lies interior to the domain

$$|\eta| \le c \xi^{\gamma_1} + d \xi^{\gamma_2}, \quad \textit{where} \quad \lambda = \xi + i\eta,$$

$$\delta > 0,\ 0 < \gamma_1,\ \gamma_2 < 1,\ m_{11} > 0,\ c > 0,\ d > 0.$$

Note the following necessary cases:

a) in papers by S. Mizohata, even in bounded domains with a cone condition, it is assumed the condition

$$\left|\operatorname{Im}\lambda_j\right| < \varepsilon\left|\operatorname{Re}\lambda_j\right|;$$

b) dense definiteness of the considered bilinear form, its closeness and spectral property require more rigid conditions on the coefficients of the operator L;

c) these theorems are proved by the method of papers [4], [7], [9].

Theorem 3. *By fulfilling the conditions of Theorem 2, there exists a number* $r > \dfrac{n-1}{2b}$, *that*

$$\sum_{j=1}^{\infty}\left|\frac{1}{(\lambda-\lambda_j)^r} - \frac{1}{(\lambda-\xi_j)^r}\right| = O(1)tr\int_{\Omega} F(x)[\lambda - l_1(x)]^{\frac{n}{2b}-r}\,dx, \tag{1.18}$$

where

$$F(x) = \int_{R^n}\exp\{-L_0(x,\xi)\}d\xi.$$

Theorem 4. *Let* $N(\lambda) = \sum_{\xi_j<\lambda} 1$ *and the conditions of previous theorems be fulfilled. Then*

$$N(\lambda) \sim \left[(2\pi)^n\Gamma\left(\frac{n}{2b}+1\right)\right]^{-1} tr \int_{\left\{\substack{x\in\Omega \\ l_1(x)<\lambda}\right\}} F(x)[\lambda - l_1(x)]^{\frac{n}{2b}}dx \tag{1.19}$$

Proof scheme for theorems 1-4.

The resolvent kernel of the operator L is given by the formula

$$K(x,y\lambda) = \int_0^{\infty} e^{-\lambda t}G(x,t,t)dt,$$

or, the kernel of the integral operator $(L+\lambda r)^{-u}$ has the form

$$K_u(x,y,\lambda) = \frac{1}{\Gamma(u)}\int_0^{\infty} t^{u-1}e^{-\lambda t}G(x,y,t)dt, \quad u > 1$$

and the operator is bounded. Then, if

$$\theta(x,y,\lambda) = \sum_{\xi_j<\lambda}\varphi_j(x)\overline{\varphi}_j(y)$$

the spectral function of the operator L,

$$G(x,y,z) = \int_0^{\infty} e^{-\lambda t}d\theta(x,y,t);$$

$$\int_{\Omega}\theta(x,x\lambda)dx = \sum_{\xi_j<\lambda}\int_{\Omega}\left|\varphi_j(x)\right|^2 dx = N(\lambda);$$

$$tr\int_{\Omega} G(x,x,t)dt = \int_0^{\infty} e^{-\lambda t}dN(t);$$

$$\sum_{j=1}^{\infty}\frac{1}{\left|\lambda+\xi_j\right|^u} = \int_{\Omega} K_u(x,x,\lambda)dx = \Gamma\left(u - \frac{n}{2b}\right)\left[(2\pi)^n\Gamma(u)\right]^{-1} tr\int_{\Omega} F(x)[\lambda - l_1(x)]^{\frac{n}{2b}-1}dx.$$

In formula (1.19) "mean" coefficients of differential expression as in [7] $\Omega = R^n$ (self- adjoint case) do not participate.

2. Now consider the case "equivalence" of all coefficients of a differential expression.

Introduce the following notations:

$$J = \int_{R^n} \| L(x,\varepsilon) + \lambda r \|^{-1} d\xi, \tag{2.1}$$

$$m(\lambda, y) = \inf_{\xi \in R^n} \frac{\| L(y,\xi) + \lambda r \|}{\| \nabla L(y,\xi) \|}, \tag{2.2}$$

$$G_\lambda(x,y,\vartheta) = \frac{1}{(2\pi)^n} \int_{R^n} [L(\vartheta,\xi) + \lambda r]^{-1} e^{i(\xi, x-y)} d\xi \tag{2.3}$$

and let $\inf_{y \in \Omega} m(\lambda, y) > 0$.

Theorem 5. *Let the following conditions be fulfilled:*

1) *integral (2.1) converges;*
2) $\int_\Omega m^{-p}(\lambda, \vartheta) d\vartheta < +\infty$, *for all* $\lambda > 0$, *where* $p > 0$;
3) $|a_\alpha(x) - a_\alpha(y)| \le b_\alpha(x) \|x - y\|$ *for* $|\alpha| = 2b$, $\|x - y\| \le \frac{\|x\| + 1}{2}$, $b_\alpha(x) << e^{|x|}$;
4) $|a_\alpha(x) - a_\alpha(y)| \le (1 + \|y\|) \|x - y\|^{c_\alpha}$ *for* $\|x - y\| \ge \frac{\|x\| + 1}{2}$, $c_\alpha > 0$;
5) $m_{11}(1 + \|x\|)^{\gamma_2} \le |a_\alpha(x)| \le m_{12}(1 + \|x\|)^{\gamma_2}$; $|\alpha| \le 2b - n - 1$, $m_i > 0$.

Then

$$i) \sum_{j=1}^{\infty} \frac{1}{|\lambda + \lambda_j|^2} = \sum_{j=1}^{\infty} \frac{1}{(\lambda + \xi_j)^2} + O(1) \quad for \quad \lambda \to +\infty, \tag{2.4}$$

$$ii) \sum_{j=1}^{\infty} \frac{1}{(\lambda + \xi_m)^2} \sim \int_\Omega \int_{R^n} |G_\lambda(x,y,y)|^2 dx dy \quad for \quad \lambda \to +\infty, \tag{2.5}$$

$$iii) |G_\lambda(x,y,y)| \le m_{13} \exp\left\{ -\frac{m(\lambda,y)\|x - y\|}{m_{14}} \right\}, \tag{2.6}$$

where $m_{14} > 0$ *is a sufficiently large number.*

The proof of these statements is carried out by the method of papers [7] and [9].

It follows from (2.6) that

$$\sum_{j=1}^{\infty} \frac{1}{(\mu + \xi_m)^2} \sim m_{15} \int_\Omega \frac{dy}{m^n(\mu, y)}, \tag{2.7}$$

and if it holds the representation

$$\int_\Omega \frac{dy}{m^n(\mu, y)} = \int_0^\infty \frac{d\psi(\lambda)}{(\lambda + \mu)},$$

then

$$N(\lambda) \sim m_{16} \psi(\lambda). \tag{2.8}$$

We note in conclusion that there exist a large class of examples when formulas (2.7) and (2.8) have the obvious form.

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Baimov Sh.K.
Baku State University named after M. Rasulzadeh.
23, Z.I. Khalilov str., 370148, Baku, Azerbaijan.

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