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THE MULTIPARAMETER ANALOGUE OF THE RESOLVENT OPERATOR

Abstract

Let the multiparameter system

$$\begin{cases} A_i(\lambda)x_i = 0 \\ i = 1,2,\dots,n, \quad \lambda = (\lambda_1,\dots,\lambda_n) \in C_n \end{cases}$$

be given. Operators $E_i - A_i(\lambda)$ *act in separable Hilbert spaces* $\mathcal{H}_i$, *completely continuous, polynomially depend on* $\lambda_1,\dots,\lambda_n$.

The multiparameter analogue of the resolvent operator is introduced. It is proved the form of the expansion of the general part of this resolvent operator in the neighbourhood of the isolated eigenvalue of the (1).

Let the multiparameter system

$$\begin{cases} A_i(\lambda)x_i = 0 \\ i = 1,2,\dots,n; \quad \lambda = (\lambda_1,\dots,\lambda_n) \in C_n \end{cases} \tag{1}$$

be given, where $A_i(\lambda)$ are bounded, polynomially depending on λ operators acting on a separable Hilbert space $\mathcal{H}_i$. $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n$ is a tensor space of spaces $\mathcal{H}_i$.

Denote by $A_i^+(\lambda)$ linear operators on $\mathcal{H}$, induced by the operator $A_i(\lambda)$ by the following way: on the decomposable tensor $x = x_1 \otimes \dots \otimes x_n \in \mathcal{H}$ we have

$$A_i^+(\lambda)x = x_1 \otimes \dots \otimes x_{i-1} \otimes A_i(\lambda)x_i \otimes \dots \otimes x_n,$$

and in all other elements of the space $\mathcal{H}$, the operator $A_i^+(\lambda)$ is defined on linearity and continuity.

$\lambda^0 = (\lambda_1^0,\dots,\lambda_n^0) \in C^n$ is the eigenvalue of (1), if there exist nonzero elements $x_{0i} \in \mathcal{H}_i$ such that

$$A_i(\lambda^0)x_{0i} = 0, \quad i = 1,2,\dots,n$$

are fulfilled.

Decomposable tensor $x_{0\dots0} = x_{01} \otimes x_{02} \otimes \dots \otimes x_{0n} \in \mathcal{H}$ is the eigenelement of (1), responding to the eigenvalue λ^0.

Denote by σ_p a set of all eigenvalues of (1).

The tensor $x_{m_1,\dots,m_n}$ is the $(m_1,m_2,\dots,m_n)$-th associated to the eigenelement $x_{0\dots0}$ with eigenvalue $\lambda^0 = (\lambda_1^0,\dots,\lambda_n^0)$, if for all $i = 1,2,\dots$ the following equalities are fulfilled

$$A_i^+(\lambda^0)x_{0\dots0} = 0$$

$$A_i^+(\lambda^0)x_{10\dots0} + \frac{1}{1!}\frac{\partial}{\partial\lambda_1}A_i^+(\lambda^0)x_{0\dots0} = 0$$

....

$$A_i^+(\lambda^0)x_{0\dots01} + \frac{1}{1!}\frac{\partial}{\partial\lambda_n}A_i^+(\lambda^0)x_{0\dots0} = 0$$

$$A_i^+(\lambda^0)x_{110...0}+\frac{1}{1!}\frac{\partial}{\partial\lambda_1}A_i^+(\lambda^0)x_{010...0}+\frac{1}{1!}\frac{\partial}{\partial\lambda_2}A_i^+(\lambda^0)x_{10...0}+$$

$$+\frac{1}{1!\,1!}\frac{\partial^2 A_i^+(\lambda^0)}{\partial\lambda_1\partial\lambda_2}x_{0...0}=0 \qquad (2)$$

....

$$\sum_{0\le r_s\le k_s}\frac{1}{r_1!\dots r_n!}\frac{\partial^r A_i^+(\lambda^0)}{\partial\lambda_1^{r_1}\cdots\partial\lambda_n^{r_n}}x_{k_1-r_1,\dots,k_n-r_n}=0$$

$$k_s\le m_s,\quad s=\overline{1,n}$$

$$\sum_{0\le i_s\le m_s}\frac{1}{i_1!\cdots i_n!}\frac{\partial^{i_1+\dots+i_n}}{\partial\lambda_1^{i_1}\dots\partial\lambda_n^{i_n}}A_i^+(\lambda)x_{m_1-i_1,\dots,m_n-i_n}=0$$

$(k_1,\dots,k_n)$, $k_i\le m_i$, $i=\overline{1,n}$ is some arrangement from the set of entire non-negative members on n with possible repetitions and zeros.

The presence of zero in the arrangement $(s_1,\dots,s_n)$, corresponding to the associated element $x_{s_1s_2\dots s_n}=x$, for instance $s_k=0$ means that in presentation of the associated element differentiation by λ_k is absent.

We can convince oneself that by virtue of continuity of all derivatives on parameters, the sequential differentiation order is indifferent and it makes possible by denotation of the mixed derivative to collect all differentiations on the same variable.

The number of all equations from (2) having the solution we call the multiplicity of the eigenelement $x_{0...0}$.

With each associated element $x_{i_1i_2\dots i_n}$ we connect some direction $\alpha(x_{i_1i_2\dots i_n})=$ $=\alpha(i_1,i_2,\dots,i_n)=(\beta(i_1),\dots,\beta(i_n))$, where $\beta(i)=0$ for $i=0$ and $\beta(i)=1$ for $i\ne 0$.

Consequently, each direction $\alpha(x_{i_1i_2\dots i_n})$ is a vector from R^n with coordinates 0 and 1. Directions $\alpha(0,0,\dots,i_s,0,\dots,0)$ denote by α_s $(s=1,2,\dots,n)$, where s is the member of vector coordinates. The greatest member of linearly independent associated elements in direction α_s we denote by m_s.

The quantity m_s+1 we call the multiplicity of the eigenvalue λ^0 in direction α_s.

Multiplicities of associated elements in all possible directions $\alpha(i_1,\dots i_n)$ are introduced similarly.

We say that the set

$$\{x_{i_1,i_2,\dots,i_n}\}_{0\le i_s\le m_s}\qquad (s=\overline{1,n})$$

is the chain of associated elements $\left(\sum_1^n i_k>0\right)$ to the eigenelement $\left(\sum_1^n i_k=0\right)$. Following M.V.Keldysh under a canonic system of eigen and associated (i.a.) elements for $\lambda^0=(\lambda_1^0,\dots,\lambda_n^0)$ we understand the system

$$\{x_{i_1i_2\dots i_n}^{(k)}\}_{0\le i_s\le m_s},\quad s=1,2,\dots,n \qquad (3)$$

possessing the following properties:

a) elements $x_{0...0}^{(k)}$ form a basis of its own subspace $M(\lambda^0)$;

b) $x^{(1)}_{0\ldots0}$ is an eigen element, whose multiplicity achieves the possible maximum p_1+1 in all directions $\alpha(s_1,\ldots,s_n)$;

c) $x^{(k)}_{0\ldots0}$ is an eigen element not expressed linearly by $x^{(1)}_{0\ldots0},\ldots,x^{(k-1)}_{0\ldots0}$, whose sum of multiplicities in all directions achieves the possible maximum p_k+1;

d) elements (3) form a chain of associated elements.

The product $\bigcup_1^n(p_s+1)$ we call the multiplicity of the eigenvalue λ^0 in possible directions. The space stretched on elements from (3) forms root subspace of system (1) corresponding to the eigenvalue λ^0. In the case of one equation and one parameter, the definition coincides with definition in [1].

Now we give some analogy of the resolvent for our system (1).

The problem on the existence of the resolvent for the operator A is closely connected with the existence of the inverse operator $A-\lambda E\ (\lambda\in C)$, that exists if and only if λ is not the eigenvalue of the operator A. In addition, the inverse operator may be both bounded and unbounded.

If the inverse operator exists and bounded, the λ is the point of the resolvent set $\lambda\in p(A)$.

In multiparametric case the matter is some otherwise. Even if $\lambda^0\in C^n$ is not an eigenvalue of system (1), nevertheless by the given system from n of elements $y_1,\ldots,y_n$, where $y_i\in R_{A_i(\lambda^0)}$, it is not always succeeded to reestablish preimage of this system, i.e. λ^0 may be an eigenvalue even if of one equation from (1).

Now show that if λ^0 is not an eigenvalue of (1), it is possible to reestablish the tensor $x_1\otimes x_2\otimes\ldots\otimes x_n$ by a unique way.

For this purpose, define some analogy of a resolvent for our system (1).

By means of system (1) construct the operator $A^+(\lambda)$, acting from the space $\mathcal{H}=\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n$ to the space $\mathcal{H}^n=\overset{n}{\underset{1}{\oplus}}(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n)$ composed of elements that may be presented in the form of linear combinations of elements of the following set

$$\{y_1\otimes x_2\otimes\ldots\otimes x_n, x_1\otimes y_2\otimes x_3\otimes\ldots\otimes x_n,\ldots,x_1\otimes x_2\otimes\ldots\otimes y_n\}. \tag{4}$$

Denote by the M the space, stretched on these linear combinations. We have from (4)

$$A^+(\lambda)\sum_1^s x_1^i\otimes\ldots\otimes x_n^i=\left(A_1^+(\lambda)\sum_1^s x_1^i\otimes\ldots\otimes x_n^i,\ldots,A_n^+(\lambda)\sum_1^s x_1^i\otimes\ldots\otimes x_n^i\right)=$$

$$=\left(\sum_{i=1}^s y_1^i\otimes x_2^i\otimes\ldots\otimes x_n^i,\ldots,\sum_{i=1}^s x_1^i\otimes x_2^i\otimes\ldots\otimes y_n^i\right), \tag{5}$$

where $y_k^i=A_k(\lambda)x_k^i$, $i=1,2,\ldots,s$; $k=1,2,\ldots,n$.

Let $\lambda\notin\sigma_p$, then λ is not an eigenvalue even if one of operators $A_i(\lambda)$, that is sufficient for reestablishment of the element $\sum_{i=1}^s x_1^i\otimes\ldots\otimes x_n^i$.

If $\lambda \in \sigma_p$, then the kernel $A^+(\lambda)$ is not zero, $0 \neq \bar{x} \in \ker A^+(\lambda)$, $\left[A^+(\lambda)\right]^{-1}$ doesn't exist and conversely, from the fact that $\ker\left(A^+(\lambda)\right)$ is not zero, it follows that λ is an eigenvalue of $A^+(\lambda)$.

Consequently, $\left(A^+(\lambda)\right)^{-1}$ exists if and only if λ is not an eigenvalue of (1).

Under the resolvent of system (1) we understand the operator acting from $\mathscr{H}^n$ to M and satisfying the condition

$$
\begin{aligned}
R^+(\lambda)A^+(\lambda) &= E, \\
A^+(\lambda)R^+(\lambda) &= \overline{E},
\end{aligned}
\tag{6}
$$

where E and $\overline{E}$ are unit operators in $\mathscr{H}$ and M correspondingly.

Theorem. *Let all operators of $A_i(\lambda) - E_i$ be completely continuous $(i = 1,2,\ldots,n)$, $\lambda^0 = \left(\lambda_1^0,\ldots,\lambda_n^0\right)$ is the isolated eigenvalue of (1). Then the main part of resolvent expansion of system (1) in the neighbourhood of the isolated eigenvalue λ^0 has poles in each direction α_i of order $m_1 + 1$ and may be represented as*

$$
\sum_{r=1}^{n}\sum_{k}\sum_{0\le s_k\le m_{k_r}} \frac{\sum\limits_{0\le i_r\le s_r} y^{(k)}_{0,\ldots,i_r,0,\ldots,0}\,\overline{z}_{0,0,\ldots,0,s_r-i_r,0,\ldots,0}}{\left(\lambda_r - \lambda_r^0\right)^{m_{k_r}-s_r+1}}, \qquad \lambda_r \neq \lambda_r^0 , \tag{7}
$$

where $m_{k_r} + 1$ is the multiplicity of the eigenelement $y^{(k)}_{00\ldots0}$ in direction α_r, and $y^{(k)}_{0\ldots0}$, $y^{(k)}_{0,\ldots,0,1,0,\ldots,0}, \ldots, y^{(k)}_{0,\ldots,0,m_{k_r},0,\ldots,0}$ is a system of eigen and associated elements of system (1) in direction α_r.

If $y \in \mathscr{H}$, $\bar{z} \in M$, under the expression $y\bar{z}$ we understand the operator B acting from M to $\mathscr{H}$, which is defined on the element $\bar{f} \in M$ by the rule $B\bar{f} = \left[\bar{f}, \bar{z}\right]y$.

Proof. Let λ^0 be an eigenvalue of system (1), then $\lambda^0 = \left(\lambda_1^0,\ldots,\lambda_n^0\right)$ is also an eigenvalue of the operator $A_1(\lambda)$. Consider the points of the form $\left(\lambda_1, \lambda_2^0 \ldots, \lambda_n^0\right)$ from the neighbourhood of the eigenvalue λ^0. The operator $A_1\left(\lambda_1, \lambda_2^0 \ldots, \lambda_n^0\right)$ under the fixed $\lambda_2 = \lambda_2^0, \ldots, \lambda_n = \lambda_n^0$ depends on one parameter, since by the condition of the Theorem $A_1^+\left(\lambda_1, \lambda_2^0 \ldots, \lambda_n^0\right)$ it has a discrete spectrum. Use the expansion of the resolvent of the operator $A^+\left(\lambda_1, \lambda_2^0 \ldots, \lambda_n^0\right)$ in the neighbourhood of the point $\left(\lambda_1, \lambda_2^0 \ldots, \lambda_n^0\right)$ under the fixed $\lambda_2^0 \ldots, \lambda_n^0$.

This expansion has the form of [1]

$$
R\left(\lambda_1, \lambda_2^0 \ldots, \lambda_n^0\right)\bar{x} = \frac{R_0^{(1)}\bar{x}}{\left(\lambda_1 - \lambda_1^0\right)^{m_1+1}} + \cdots + \frac{R_{m_k}^{(1)}\bar{x}}{\lambda_1 - \lambda_1^0} + \widetilde{R}\left(\lambda_1, \lambda_2^0, \ldots, \lambda_n^0\right)\bar{x}, \tag{8}
$$

where $\widetilde{R}\left(\lambda_1, \lambda_2^0, \ldots, \lambda_n^0\right)$ is analytic at the point $\lambda_1 = \lambda_1^0$, $R_i^{(1)}$ are finite-dimensional operators, $\bar{x} \in \mathscr{H}^n$.

Further, since the set of regular points of the system $A^+(\lambda)$ is an open set, there exists some neighbourhood of the point $\left(\lambda_1, \lambda_2^0 \ldots, \lambda_n^0\right)$, namely, the neighbourhood $\vartheta_1\left(\lambda_1, \lambda_2^0 \ldots, \lambda_n^0\right)$ of points for which the resolvent exists.

Therefore, the expansion $R(\lambda_1,\lambda_2,\dots,\lambda_n)$ in the neighbourhood ϑ_1 of the point $(\lambda_1,\lambda_2^0\dots,\lambda_n^0)$ has the form

$$R(\lambda_1,\lambda_2,\dots,\lambda_n)\overline{x}=R(\lambda_1,\lambda_2^0,\dots,\lambda_n^0)\overline{x}+$$
$$+\sum\frac{(\lambda_2-\lambda_2^0)^{i_2}\cdots(\lambda_n-\lambda_n^0)^{i_n}}{i_2!\dots i_n!}\frac{\partial^{i_2+\dots+i_n}}{\partial\lambda_2^{i_2}\dots\partial\lambda_n^{i_n}}R(\lambda_1,\lambda_2,\dots,\lambda_n)\Big|_{\substack{\lambda_i=\lambda_i^0\\ i=\overline{2,n}}}. \quad (9)$$

By substituting the expression for $R(\lambda_1,\lambda_2^0,\dots,\lambda_n^0)$ from (8) to (9) we get

$$R(\overline{\lambda}_1,\lambda_2^0\dots,\lambda_n^0)\overline{x}=\frac{R_0^{(1)}\overline{x}}{(\overline{\lambda}_1-\lambda_1^0)^{m_{k_1}+1}}+\dots+\frac{R_{m_{k_1}}^{(1)}\overline{x}}{\overline{\lambda}_1-\lambda_1^0}+\widetilde{R}(\overline{\lambda}_1,\lambda_2^0,\dots,\lambda_n^0)\overline{x}+$$
$$+\sum\frac{(\lambda_2-\lambda_2^0)^{i_2}\cdots(\lambda_n-\lambda_n^0)^{i_n}}{i_2!\dots i_n!}\frac{\partial^{i_2+\dots+i_n}}{\partial\lambda_2^{i_2}\dots\partial\lambda_n^{i_n}}R(\lambda_1,\lambda_2,\dots,\lambda_n)\overline{x}\Big|_{\substack{\lambda_k=\lambda_k^0\\ k=\overline{2,n}}},\quad \lambda\in\vartheta_1. \quad (10)$$

The latter is valid for all $(\lambda_1,\lambda_2^0\dots,\lambda_n^0)$ for which (8) is valid. We can write similar for the neighbourhood ϑ_2 of the point $(\lambda_1^0,\lambda_2,\lambda_3^0\dots,\lambda_n^0)$ or rather for such points from ϑ_2, for which (11) is valid, have the second coordinate $\lambda_2=\lambda_2^0$.

Indeed,

$$R(\lambda_1,\lambda_2^0\dots,\lambda_n^0)\overline{x}=\frac{R_0^{(2)}\overline{x}}{(\lambda_2-\lambda_2^0)^{m_{k_2}+1}}+\dots+\frac{R_{m_{k_2}}^{(2)}\overline{x}}{\lambda_2-\lambda_2^0}+\widetilde{R}(\lambda_1^0,\lambda_2,\lambda_3^0,\dots,\lambda_n^0)\overline{x}+$$
$$+\sum\frac{(\lambda_1-\lambda_1^0)^{i_1}\cdots(\lambda_n-\lambda_n^0)^{i_n}}{i_1!i_3!\dots i_n!}\frac{\partial^{i_1+i_3+\dots+i_n}}{\partial\lambda_1^{i_1}\partial\lambda_3^{i_3}\dots\partial\lambda_n^{i_n}}R(\lambda_1,\lambda_2,\dots,\lambda_n)\overline{x}\Big|_{\lambda_i=\lambda_i^0}. \quad (11)$$

For arbitrary n

$$R(\lambda_1,\lambda_2\dots,\lambda_n)\overline{x}=\frac{R_0^{(n)}\overline{x}}{(\overline{\lambda}_n-\lambda_n^0)^{m_{k_n}+1}}+\dots+\frac{R_{m_{k_n}}^{(n)}\overline{x}}{\overline{\lambda}_n-\lambda_n^0}+\widetilde{R}(\lambda_1^0,\lambda_2^0,\dots,\lambda_{n-1}^0,\lambda_n)\overline{x}+$$
$$+\sum\frac{(\lambda_1-\lambda_1^0)^{i_1}\cdots(\lambda_{n-1}-\lambda_{n-1}^0)^{i_{n-1}}}{i_1!\dots i_{n-1}!}\frac{\partial^{i_1+\dots+i_{n-1}}}{\partial\lambda_1^{i_1}\dots\partial\lambda_{n-1}^{i_{n}}}R(\lambda_1,\lambda_2,\dots,\lambda_n)\overline{x}\Big|_{\substack{\lambda_i=\lambda_i^0\\ i=\overline{1,n-1}}},\quad i=\overline{1,n-1}. \quad (12)$$

Now, let $(\lambda_1,\lambda_2\dots,\lambda_n)$ be some point from the neighbourhood $(\lambda_1^0,\lambda_2^0\dots,\lambda_n^0)$ in which the resolvent exists ($\lambda_i\neq\lambda_i^0$ for all i).

Then we have

$$\begin{aligned}R(\lambda_1,\dots,\lambda_n)&=\frac{1}{n}\{nR(\lambda_1,\dots,\lambda_n)-R(\lambda_1^0,\dots,\lambda_n)-R(\lambda_1,\lambda_2^0,\dots,\lambda_n)-\dots-\\&-R(\lambda_1,\lambda_2,\dots,\lambda_{n-1},\lambda_n^0)+R(\lambda_1^0,\lambda_2,\dots,\lambda_n)+R(\lambda_1,\lambda_2^0,\dots,\lambda_n)+\dots+\\&+R(\lambda_1,\lambda_2,\dots,\lambda_{n-1},\lambda_n^0)\}=\frac{1}{n}\{[R(\lambda_1^0,\dots,\lambda_n)+R(\lambda_1,\lambda_2^0,\dots,\lambda_n)+\dots+\\&+R(\lambda_1,\lambda_2,\dots,\lambda_{n-1},\lambda_n^0)]+\sum_{k=1}^{n}[R(\lambda_1,\dots,\lambda_n)-R(\lambda_1,\dots,\lambda_k^0,\dots,\lambda_n)]\}.\end{aligned} \quad (13)$$

The operator

$$\sum_{k=1}^{n}(R(\lambda_1,\dots\lambda_n)-R(\lambda_1,\dots,\lambda_k^0,\dots\lambda_n))=[R(\lambda_1,\dots,\lambda_n)-R(\lambda_1^0,\lambda_2,\dots,\lambda_n)]+$$

$$+\left[R(\lambda_1,\dots,\lambda_n)-R(\lambda_1,\lambda_2^0,\dots,\lambda_n)\right]+\dots+\left[R(\lambda_1,\lambda_2,\dots,\lambda_n)-R(\lambda_1,\lambda_2,\dots,\lambda_n^0)\right].$$

Each operator in square brackets of the last expression by virtue of the choice $\lambda_1,\lambda_2,\dots,\lambda_n$ may be arbitrarily small by the norm at the expense of the choice of the neighbourhood of the point $(\lambda_1^0,\lambda_2^0\dots,\lambda_n^0)$. Further, we have for the operator $R(\lambda_1^0,\lambda_2,\dots,\lambda_n)$

$$R(\lambda_1^0,\lambda_2,\dots,\lambda_n)=\frac{1}{n-1}\{R(\lambda_1^0,\lambda_2^0,\lambda_3,\dots,\lambda_n)+R(\lambda_1^0,\lambda_2,\lambda_3^0,\dots,\lambda_n)+\dots+$$
$$+R(\lambda_1^0,\lambda_2,\dots,\lambda_{n-1},\lambda_n^0)\}+\sum_{k=2}^{n}\left(R(\lambda_1^0,\lambda_2,\dots,\lambda_n)-R(\lambda_1^0,\lambda_2,\dots,\lambda_k^0,\lambda_n)\right). \qquad (14)$$

Similarly,

$$R(\lambda_1^0,\dots,\lambda_k^0,\lambda_{k+1},\dots,\lambda_n)=\frac{1}{n-k}\{R(\lambda_1^0,\dots,\lambda_{k+1}^0,\lambda_{k+2},\dots,\lambda_n)+$$
$$+R(\lambda_1^0,\dots,\lambda_k^0,\lambda_{k+1},\lambda_{k+2}^0,\dots,\lambda_n)+\dots+R(\lambda_1^0,\dots,\lambda_k^0,\lambda_{k+1},\dots,\lambda_{n-1},\lambda_n^0)+$$
$$+\sum_{i=k+1}^{n}\left[R(\lambda_1^0,\lambda_2^0,\dots,\lambda_k^0,\lambda_{k+1},\dots,\lambda_n)-R(\lambda_1^0,\lambda_2^0,\dots,\lambda_k^0,\lambda_{k+1},\dots,\lambda_n^0,\dots,\lambda_n)\right]\}. \qquad (15)$$

By substituting in (13) the expressions for $R(\lambda_1,\dots,\lambda_k^0,\dots,\lambda_n)$ from (14) we get

$$R(\lambda_1,\dots,\lambda_k,\dots,\lambda_n)=\frac{1}{n(n-1)}\{R(\lambda_1^0,\lambda_2^0,\lambda_3,\dots,\lambda_n)+R(\lambda_1^0,\lambda_2,\lambda_3^0,\dots,\lambda_n)+\dots+$$
$$+R(\lambda_1^0,\lambda_2,\dots,\lambda_{n-1},\lambda_n^0)+\sum_{k=2}^{n}\left[R(\lambda_1^0,\lambda_2,\dots,\lambda_n)-R(\lambda_1^0,\lambda_2,\dots,\lambda_k^0,\lambda_{n-1},\lambda_n)\right]+$$
$$+\left[R(\lambda_1^0,\lambda_2^0,\dots,\lambda_n)+R(\lambda_1^0,\lambda_2^0,\lambda_3^0,\lambda_4,\dots,\lambda_n)+R(\lambda_1^0,\lambda_2^0,\lambda_3,\dots,\lambda_n^0)\right]+$$
$$\sum_{k\neq 2}^{n}\left[R(\lambda_1,\lambda_2^0,\lambda_3,\dots,\lambda_n)-R(\lambda_1,\lambda_2^0,\dots,\lambda_k^0,\dots,\lambda_n)\right]+\left[R(\lambda_1^0,\lambda_2,\lambda_3^0,\dots,\lambda_n)+\right.$$
$$\left.+R(\lambda_1,{}_2^0,\lambda_3^0,\dots,\lambda_n)+\dots+R(\lambda_1,\lambda_2,\lambda_3^0,\lambda_4^0,\dots,\lambda_n)+R(\lambda_1,\lambda_2,\lambda_3^0,\dots,\lambda_{n-1},\lambda_n^0)\right]+ \qquad (16)$$
$$+\sum_{\substack{k=1\\k\neq 3}}^{n}\left[R(\lambda_1,\lambda_2,\lambda_3^0,\dots,\lambda_n)-R(\lambda_1,\lambda_2,\lambda_3^0,\dots,\lambda_k^0,\dots,\lambda_n)\right]+\dots+R(\lambda_1^0,\lambda_2,\dots,\lambda_n^0)+$$
$$+R(\lambda_1,\lambda_2^0,\lambda_3,\dots,\lambda_n^0)+\dots+R(\lambda_1,\lambda_2,\dots,\lambda_{n-1}^0,\lambda_n^0)+\sum_{k=1}^{n-1}\left[R(\lambda_1,\lambda_2,\dots,\lambda_n^0)-\right.$$
$$\left.-R(\lambda_1,\dots,\lambda_i^0,\dots,\lambda_n^0)\right]\}+\frac{1}{n}\sum_{k=1}^{n}\left[R(\lambda_1,\lambda_2,\dots,\lambda_n)-R(\lambda_1,\dots,\lambda_k^0,\dots,\lambda_n)\right].$$

By substituting in (16) the values for $R(\lambda_1,\lambda_2,\dots,\lambda_i^0,\dots,\lambda_k^0,\dots,\lambda_n)$ for $(i\neq k;$ $k,i=1,2,\dots,n)$ from (15), when $k=2$ we get

$$R(\lambda_1,\dots,\lambda_n)=\frac{1}{n(n-1)(n-2)}\{\left[R(\lambda_1^0,\lambda_2^0,\lambda_3^0,\dots,\lambda_n)+R(\lambda_1^0,\lambda_2^0,\lambda_3,\lambda_4^0,\dots,\lambda_n)+\dots+\right.$$
$$\left.+R(\lambda_1^0,\lambda_2^0,\lambda_3,\dots,\lambda_n^0)\right]+\sum_{k\neq 1,2}^{n}\left[R(\lambda_1^0,\lambda_2^0,\lambda_3,\dots,\lambda_n)-R(\lambda_1^0,\lambda_2^0,\lambda_3,\dots,\lambda_k^0,\dots,\lambda_n)\right]+$$
$$+R(\lambda_1^0,\lambda_2^0,\lambda_3^0,\lambda_4,\dots,\lambda_n)+R(\lambda_1^0,\lambda_2,\lambda_3^0,\lambda_4^0,\dots,\lambda_n)+\dots+R(\lambda_1^0,\lambda_2,\lambda_3^0,\lambda_4,\dots,\lambda_n^0)+ \qquad (17)$$
$$+\sum_{k\neq 1,3}^{n}\left[R(\lambda_1^0,\lambda_2,\lambda_3^0,\dots,\lambda_n)-R(\lambda_1^0,\lambda_2^0,\lambda_3^0,\dots,\lambda_k^0,\dots\lambda_n)\right]+\dots\}+\frac{1}{n(n-1)}\times$$

$$\sum_{\substack{k=i\\ i\neq k}}^{n}\left[R\left(\lambda_1,\lambda_2^0,\ldots,\lambda_n\right)-R\left(\lambda_1^0,\lambda_2^0,\ldots,\lambda_k^0,\ldots,\lambda_n\right)\right]+\frac{1}{n}\sum_{k=1}^{n}\left[R(\lambda_1,\ldots\lambda_n)-R\left(\lambda_1,\ldots,\lambda_k^0,\ldots\lambda_n\right)\right].$$

By substituting in (17) the expressions for $R\left(\lambda_1,\ldots,\lambda_{k_1}^0,\ldots,\lambda_{k_2}^0,\ldots,\lambda_{k_3}^0,\ldots,\lambda_n\right)$ from $1\leq k_1,k_2,k_3\leq n, k_1\neq k_2\neq k_3$ from (15) we get for $R(\lambda_1,\ldots,\lambda_n)$ expressions by

$$R\left(\lambda_1^0,\lambda_2^0,\lambda_3^0,\lambda_4^0,\ldots,\lambda_n\right),\ldots,R\left(\lambda_1,\ldots,\lambda_{k_1},\ldots,\lambda_{k_2},\ldots\lambda_{k_3},\ldots,\lambda_{k_4},\ldots,\lambda_n\right).$$

Consequently, by substituting expressions for $R\left(\lambda_1,\ldots,\lambda_{k_1}^0,\ldots,\lambda_{k_s}^0,\ldots,\lambda_n\right)$ from (15) we at last get expressions for $R(\lambda_1,\ldots,\lambda_n)$ by means of $R\left(\lambda_1^0,\ldots,\lambda_{n-1}^0,\lambda_n\right),\ldots,$ $R\left(\lambda_1,\lambda_2^0,\ldots,\lambda_n^0\right)$ and finite number of sums the norms of which may be arbitrary small at expense of the choice of numbers $\lambda_1,\lambda_2,\ldots,\lambda_n$ chosen from the neighbourhood of the point $\left(\lambda_1^0,\lambda_2^0,\ldots,\lambda_n^0\right)$.

Thus, representation of the resolvent $R(\lambda_1,\ldots,\lambda_n)$ at the neighbourhood of the point $\lambda^0=\left(\lambda_1^0,\ldots,\lambda_n^0\right)$, excluded the points for which even if one of $\lambda_i=\lambda_i^0$ $\left(i=\overline{1,n}\right)$ is the sum of expansions of resolvents $R\left(\lambda_1^0,\lambda_2^0,\ldots,\lambda_{k-1}^0,\lambda_k,\lambda_{k+1}^0,\ldots,\lambda_{n-1}^0,\lambda_n^0\right)$ at the neighborhood of the point $\left(\lambda_1^0,\ldots,\lambda_n^0\right)$ and the operators small on the norm.

The principal part of the expansion of the resolvent $R\left(\lambda_1^0,\lambda_2^0,\ldots,\lambda_{r-1}^0,\lambda_r,\lambda_{r+1}^0,\ldots,\lambda_{n-1}^0,\lambda_n^0\right)$ at the neighbourhood of the point λ_r^0 according to M.V. Keldysh's theory may be represented in the form

$$\sum_k\sum_{0\leq s_k\leq m_{k_r}}\frac{\sum\limits_{0\leq i_r\leq s_r} y^{(k)}_{0,\ldots,0,i_r,0,\ldots,0}\,\overline{z}^{(k)}_{0,0,\ldots,0,s_r-i_r,0,\ldots,0}}{\left(\lambda_r-\lambda_r^0\right)^{m_{k_r}-s_r+1}},\quad \lambda_r\neq\lambda_r^0,$$

where $m_{k_r}+1$ is the multiplicity of the eigen-vector $y^{(k)}_{0,0,\ldots,0}$ in direction α_r, and

$$y^{(k)}_{0,0,\ldots,0},y_{0,0,\ldots,0,1,0,\ldots,0},\ldots,y_{0,0,\ldots,m_r,0,\ldots,0}$$

is the system of eigen and associated elements of the system (1) in direction α_r.

By summing the general parts of expansions of resolvents $R\left(\lambda_1,\lambda_2^0,\ldots,\lambda_n^0\right)$, $R\left(\lambda_1^0,\lambda_2,\lambda_3^0,\ldots,\lambda_n^0\right),\ldots,R\left(\lambda_1^0,\lambda_2^0,\ldots,\lambda_{n-1}^0,\lambda_n\right)$ where sequentially all parameters are fixed except one, we get the expansion (7) corresponding to the statement of the theorem.

The theorem has been proved.

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