GADJIEV A.D., ISPIR N.

ON A SEQUENCE OF LINEAR POSITIVE OPERATORS IN WEIGHTED SPACES

Abstract

The authors present a modified sequences of operators which were introduced by Ibragimov and Gadjiev and established the theorems on weighted approximation by these operators on infinite sets.

Keywords: *Linear positive operators, Korovkin type theorems, modulus of continuity*[1,2].

1. Introduction.

The sequences of linear positive operators

$$L_n(f,x)=\sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n^2\psi_n(0)}\right)\left(\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{\substack{u=\alpha_n\psi_n(0)\\ t=0}}\right)\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!} \qquad (1)$$

were introduced by Ibragimov and Gadjiev [1] [3] under some conditions on functions $K_n(x,t,u), \psi_n(t)$ and the sequence $\{\alpha_n\}$, which shall be noted below. In [1] authors showed that this sequence was a generalization of well-known Bernstein, Szasz, Bernstein- Chlodowsky and Baskakov operators. Some new properties of the operators (1) were established in different papers. We refer to [6], [7]. P. Radatz and B.Wood [6] have given an approximation of derivatives of functions in a certain class by the derivatives of the operators (1). Generalization of this result may be found in [7]. In [2], [3], [4] and [5] some results on the approximation of derivatives by derivatives of Baskakov type operators have been given.

Note that all of these papers, including the Gadjiev- Ibragimov's paper [1], are devoted to the case of finite interval $[0,A]$.

The aim of this paper is to solve the problem of weighted approximation of continuous and unbounded functions defined on semiaxis by the modificated operators (1), which will be defined below.

Note that the weighted Korovkin's type theorem were established in [8], [9]. We give those theorems, which will be used throughout the paper.

Let $\rho(x)=1+x^2, -\infty<x<\infty$ and B_ρ be the set of all functions f defined on the real axis satisfying the condition $|f(x)|\le M_f\rho(x)$ with some constant M_f, depending only on f. By C_ρ, we denote the subspace of all continuous functions belong to B_ρ. Also, let C_ρ^k be the subspace of all functions $f\in C_\rho$, for which $\lim\limits_{|x|\to\infty} f(x)/\rho(x)$ is finite.

Theorem A. *Let the sequence of linear positive operators* $\{L_n\}$, *acting from* C_ρ

[1] 1991 Mathematics Subject Classification 41A36
[2] Supported by Ank.Univ. 99-05-03-02-01
[3] A.D. Gadjiev = A.D. Gadziev

to B_ρ *satisfy the conditions*

$$\lim_{n\to\infty}\left\|L_n\left(t^\nu,x\right)-x^\nu\right\|_\rho=0,\quad \nu=0,1,2.$$

Then, for any function $f\in C_\rho^k$,

$$\lim_{n\to}\left\|L_nf-f\right\|_\rho=0,$$

and there exists a function $f^*\in C_\rho\setminus C_\rho^k$ *such that*

$$\overline{\lim_{n\to}}\left\|L_nf^*-f^*\right\|_\rho\geq 1.$$

Now, let $\{a_n\}$ be a sequence of positive numbers such that $\lim\limits_{n\to\infty}a_n=\infty$ and

$$\|f\|_{\rho,[0,a_n]}=\sup_{0\leq x\leq a_n}\frac{|f(x)|}{\rho(x)}.$$

For linear positive operators, acting from C_ρ to $B_{\rho,[0,a_n]}$, theorem A gives the following (see [10])

Theorem B. *The conditions*

$$\lim_{n\to\infty}\left\|L_n\left(t^\nu,x\right)-x^\nu\right\|_{\rho,[0,a_n]}=0,\quad \nu=0,1,2$$

implies

$$\lim_{n\to\infty}\left\|L_nf-f\right\|_{\rho,[0,a_n]}=0$$

for any function $f\in C_\rho^k$.

Now let us determine the modificated operators (1).

Let $\{\gamma_n\}$ be the sequence of positive numbers, which has finite or infinite limit. Let $\{K_n(x,t,u)\}$ be a sequence of functions of three variables $(x,t\in[0,\gamma_n],-\infty<u<\infty)$, having the following properties:

1^0. Every function of this sequence is an entire analytic function with respect to u for fixed x and t on $[0,\gamma_n]$;

2^0. $K_n(x,0,0)=1$ for any $x\in[0,\gamma_n]$;

3^0. $\left\{(-1)^\nu\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{u=u_1,t=0}\right\}\geq 0,\quad (\nu=0,1,2,\ldots,x\in[0,\gamma_n]);$

4^0. $\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{u=u_1,t=0}=-nx\left[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}}K_{n+m}(x,t,u)\Big|_{u=u_1,t=0}\right],$

$(x\in[0,\gamma_n];\nu,n=1,2,\ldots)$, where m is a such number that $n+m$ is zero or a natural number.

Moreover, let $\{\varphi_n(t)\}$ and $\{\psi_n(t)\}$ be two sequences of the class $C[0,\gamma_n]$ such that, $\varphi_n(0)=0$ for each $t\in[0,\gamma_n]$, $\psi_n(t)>0$ and

$$\lim_{n\to\infty}\frac{1}{n^2\psi_n(0)}=0. \tag{2}$$

Also, let $\{\alpha_n\}$ denote a sequence of positive numbers satisfying the condition

$$\frac{\alpha_n}{n}=1+O\left(\frac{1}{n^2\psi_n(0)}\right). \tag{3}$$

We call the operators (1) under the above conditions the modificated operators.

Note that in the case of $\lim\limits_{n\to\infty}\gamma_n = A$ and $\lim\limits_{n\to\infty}\frac{\alpha_n}{n}=1$ we obtain Gadjiev-Ibragimov operators, defined in [1].

Obviously, the modificated operator is a linear positive operator having the properties

$$\begin{aligned} L_n(1,x)&=1,\\ L_n(t,x)&=\frac{\alpha_n}{n}x, \\ L_n(t^2,x)&=\left(\frac{\alpha_n}{n}\right)^2\frac{n+m}{n}x^2+\frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}x. \end{aligned} \tag{4}$$

2. Convergence of the modificated operators in weighted spaces.

Theorem 1. *Let $\{L_n\}$ be the sequence of linear positive operators, defined by (1). Then for each function $f\in C_\rho^k$,*

$$\lim_{n\to\infty}\|L_nf-f\|_{\rho,[0,\gamma_n]}=0.$$

Proof. Clearly, $\|L_n(1;x)-1\|_{\rho,[0,\gamma_n]}\to 0$ as $n\to\infty$ on $[0,\gamma_n]$. The second of the equality in (4) and the condition (3) give

$$\sup_{x\in[0,\gamma_n]}\frac{|L_n(t,x)-x|}{1+x^2}\le\left|\frac{\alpha_n}{n}-1\right|\sup_{x\in[0,\gamma_n]}\frac{x}{1+x^2}=O\left(\frac{1}{n^2\psi_n(0)}\right).$$

Hence, by (3) and (2), we obtain

$$\lim_{n\to\infty}\sup_{x\in[0,\gamma_n]}\frac{|L_n(t,x)-x|}{1+x^2}=0.$$

Using last equality of (4) we can write

$$\sup_{x\in[0,\gamma_n]}\frac{\left|L_n(t^2,x)-x^2\right|}{1+x^2}\le\left|\left(\frac{\alpha_n}{n}\right)^2\frac{n+m}{n}-1\right|\sup_{x\in[0,\gamma_n]}\frac{x}{1+x^2}+\frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}\sup_{x\in[0,\gamma_n]}\frac{x}{1+x}\le$$

$$\le\left|\left(\frac{\alpha_n}{n}\right)^2\frac{n+m}{n}-1\right|+\frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)},$$

which implies that

$$\lim_{n\to\infty}\sup_{x\in[0,\gamma_n]}\frac{\left|L_n(t^2,x)-x^2\right|}{1+x^2}=0.$$

Since the conditions of theorem B are satisfied, we obtain for any $f\in C_\rho^k$

$$\lim_{n\to\infty}\sup_{x\in[0,\gamma_n]}\frac{|L_n(f,x)-f|}{1+x^2}=0,$$

which completes the proof.

Now, we investigated a rate of convergence of the sequence of the operators $\{L_n(f(t);x)$. By the definition of the operator (1) it is seen that $t\in[0,\infty)$. Thus, we can not find a rate of convergence in terms of usual first modulus of continuity $\omega(f,\delta)$ of function f because a modulus of continuity $\omega(f,\delta)$ on infinite interval does not tend to zero as $\delta \to 0$. By this reason, we consider a weighted modulus of continuity $\Omega(f,\delta)$, which has this property.

For each $f \in C_\rho^k$, let

$$\Omega(f;\delta) = \sup_{|h|\le\delta,t,x\in[0,\gamma_n]} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}.$$

We call $\Omega(f;\delta)$ the modulus of continuity of the function f on the space C_ρ^k. Consider the properties of function $\Omega(f,\delta)$. By the definition of C_ρ^k, for given $\varepsilon > 0$ we can find $x_0 = x_0(\varepsilon)$ such that the inequality

$$\left|\frac{f(x)}{1+x^2} - k_f\right| < \varepsilon$$

holds for all $x > x_0$. Therefore

$$\Omega(f,\delta) \le \sup_{0\le x\le x_0, |h|\le\delta} |f(x+h)-f(x)| + \sup_{x_0\le x\le\gamma_n} \left|\frac{f(x+h)}{1+(x+h)^2} - k_f\right| +$$

$$+\delta k_f \sup_{x_0\le x\le\gamma_n} \frac{2x+\delta}{1+x^2} + \sup_{x_0\le x\le\gamma_n} \left|\frac{f(x)}{1+x^2} - k_f\right| < \omega(f,\delta) + 2\delta k_f + \varepsilon,$$

where $\omega(f,\delta)$ is the usual first modulus of continuity of f on the interval $[0,x_0]$ and $\lim_{\delta\to 0}\omega(f,\delta) = 0$ since f is uniformly continuous on $[0,x_0]$. Consequently,

$$\lim_{\delta\to 0}\Omega(f,\delta) = 0, \quad \text{for every } f \in C_\rho^k.$$

Now for a natural number m can write

$$|f(x+mh)-f(x)| = \left|\sum_{k=1}^{m} f(x+kh) - f(x+(k-1))h\right| \le$$

$$\le (1+h^2)\sum_{k=1}^{m}\left(1+\left(x+(k-1)h^2\right)\right)\Omega(f;\delta).$$

Hence we obtain the inequality

$$\Omega(f;m\delta) \le 2m(1+\delta^2)\Omega(f;\delta)$$

and then for any $\lambda > 0$, $\Omega(f;\lambda\delta) \le 2(1+\lambda)(1+\delta^2)\Omega(f;\delta)$ also holds. This properties of modulus of continuity $\Omega(f,\delta)$ and its definition show that for every $f \in C_\rho^k[0,\infty)$

$$|f(t)-f(x)| \le (1+x^2)(1+(t-x)^2)\Omega(f;|t-x|) \le$$

$$\le 2\left(\frac{|t-x|}{\delta}+1\right)(1+x^2)(1+(t-x)^2) \tag{5}$$

for each $x,t \in [0,\gamma_n]$.

Remark 1. For the rest of these paper the paper the expression $\left.\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\right|_{u=\alpha_n\psi_n(0),t=0}$ will be denoted by $K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))$.

Theorem 2. *If $f \in C_\rho^0$, then the inequality*

$$\sup_{x\in[0,\gamma_n]} \frac{|L(f;x)-f(x)|}{(1+x^2)^3} \le K\Omega\left(f;\frac{1}{\sqrt{n^2\psi_n(0)}}\right) \tag{6}$$

holds for a sufficiently large n, where K is a constant indepent on n.

Proof. Since

$$\sum_{\nu=0}^{\infty} K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}=1$$

we get

$$L_n(f,x)-f(x)=\sum_{\nu=0}^{\infty}\left[f\left(\frac{\nu}{n^2\psi_n(0)}\right)-f(x)\right]K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}.$$

Using (5) we can write

$$|L_n(f,x)-f(x)|\le 2(1+x^2)(1+\delta_n^2)\Omega(f;\delta_n)\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\times$$

$$\times\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}\left(1+\frac{\left|x-\frac{\nu}{n^2\psi_n(0)}\right|}{\delta_n}\right)\left(1+\left(x-\frac{\nu}{n^2\psi_n(0)}\right)^2\right)\le$$

$$\le 4(1+x^2)\Omega(f;\delta_n)\left\{1+\frac{1}{\delta_n}\sum_{\nu=0}^{\infty}\left|x-\frac{\nu}{n^2\psi_n(0)}\right|K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}+\right.$$

$$+\sum_{\nu=0}^{\infty}\left(x-\frac{\nu}{n^2\psi_n(0)}\right)^2K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}+$$

$$\left.+\frac{1}{\delta_n}\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left|x-\frac{\nu}{n^2\psi_n(0)}\right|\left(x-\frac{\nu}{n^2\psi_n(0)}\right)^2\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}\right\}.$$

Applying the Hölder's inequality, we obtain

$$|L_n(f,x)-f(x)|\le 4(1+x^2)\Omega(f;\delta_n)\left(1+\frac{2}{\delta_n}I_1^{1/2}+I_1+\frac{1}{\delta_n}I_2\right), \tag{7}$$

where

$$I_1=\sum_{\nu=0}^{\infty}\left(x-\frac{\nu}{n^2\psi_n(0)}\right)^2K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}$$

and

$$I_2=\sum_{\nu=0}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\left(x-\frac{\nu}{n^2\psi_n(0)}\right)^4\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}.$$

Using the equalities (4), we obtain for I_1

$$I_1=x^2\left(1-2\left(\frac{\alpha_n}{n}\right)\frac{n+m}{n}+\left(\frac{\alpha_n}{n}\right)^2\frac{(n+m)(n+2m)}{n^2}\right)+$$

$$+x\left(\frac{\alpha_n}{n}\right)\frac{n+m}{n}\frac{1}{n^2\psi_n(0)} \tag{8}$$

and for I_2

$$I_2 = x^4\left(1-4\left(\frac{\alpha_n}{n}\right)\frac{(n+m)}{n}-4\left(\frac{\alpha_n}{n}\right)^3\frac{(n+m)(n+2m)(n+3m)}{n^3}+6\left(\frac{\alpha_n}{n}\right)^2\times\right.$$
$$\left.\times\frac{(n+m)(n+2m)}{n^2}+\left(\frac{\alpha_n}{n}\right)^4\frac{(n+m)(n+2m)(n+3m)(n+4m)}{n^4}\right)+x_3\left(6\left(\frac{\alpha_n}{n}\right)\frac{(n+m)}{n}\times\right.$$
$$\times\frac{1}{(n^2\psi_n(0))^2}-12\left(\frac{\alpha_n}{n}\right)^2\frac{(n+m)(n+2m)}{n^2}\frac{1}{(n^2\psi_n(0))^2}+6\left(\frac{\alpha_n}{n}\right)^3\frac{(n+m)(n+2m)(n+3m)}{n^3}\times$$
$$\left.\times\frac{1}{n^2\psi_n(0)}\right)+7x^2\left(\frac{\alpha_n}{n}\right)^2\frac{(n+m)(n+2m)}{n^2}\frac{1}{(n^2\psi_n(0))^2}+x\left(\frac{\alpha_n}{n}\right)\frac{(n+m)}{n}\frac{1}{(n^2\psi_n(0))^3}. \tag{9}$$

Using the condition (3), the equalities (8) and (9) can be written as follows

$$I_1 = (x^2+x)O\left(\frac{1}{n^2\psi_n(0)}\right)$$

and

$$I_2 = (x^4+x^3+x^2+x)O\left(\frac{1}{n^2\psi_n(0)}\right).$$

Using this equalities in (7), we have

$$|L_n(f,x)-f(x)|\le 4(1+x^2)\Omega(f;\delta_n)\left\{1+\frac{2}{\delta_n}\sqrt{(x^2+x)O\left(\frac{1}{n^2\psi_n(0)}\right)}+\right.$$
$$\left.+(x^2+x)O\left(\frac{1}{n^2\psi_n(0)}\right)+\frac{1}{\delta_n}(x^4+x^3+x^2+x)O\left(\frac{1}{n^2\psi_n(0)}\right)\right\}.$$

Choosing $\delta_n = \dfrac{1}{\sqrt{n^2\psi_n(0)}}$, we have

$$|L_n(f,x)-f(x)|\le 4(1+x^2)\Omega(f;\delta_n)\left\{1+2\sqrt{x^2+x}+\right.$$
$$\left.\delta_n^2(x^2+x)+(x^4+x^3+x^2+x)\right)$$

and since $\delta_n^2<1$ for a large n, we obtain the basic inequality

$$|L_n(f,x)-f(x)|\le 4(1+x^2)\Omega(f;\delta_n)\left\{1+2\sqrt{x^2+x}+x^4+x^3+2x^2+2x\right\}$$

From this, for a large n, we can write

$$\sup_{x\in[0,\gamma_n]}\frac{|L_n(f;x)-f(x)|}{(1+x^2)^3}\le K\Omega\left(f,\frac{1}{\sqrt{n^2\psi_n(0)}}\right),$$

where K is a positive constant. The proof is completed.

Remark 2. As shown in theorem A, $\{L_n(f;x)\}$ converges to $f(x)$ in weighted spaces with weight function $\rho(x)=1+x^2$. But, as may be seen from theorem B we could find the rate of convergence of the operators (1) only in weighted spaces with weighted function $(1+x^2)^3$. Therefore the rate of convergence of the operators (1) is an open problem when weighted function $(1+x^2)^\alpha$, $1\le\alpha<3$.

3. Existence of the derivatives $L_n^{(p)}(f;x)$.

Let f be $(p-1)$- times continuously differentiable function on $[0,\infty)$, which belongs to C_ρ and let also its p-th derivative $f^{(p)}$ satisfy the Lipschitz condition

$$\left|f^{(p)}(x)-f^{(p)}(t)\right|\le M|x-t|^\alpha,\quad 0<\alpha\le 1$$

for any $x,t\ge 0$. In this case, we write $f^{(p)}\in Lip_M\alpha$.

Now, supposed that the function $K_n(x,t,u)$, in addition to the conditions 1^0-4^0, satisfies

5^0. $K_n(x,t,u)$ is continuously differentiable with respect to x for any fixed u and t on the interval $[0,\gamma_n]$ and

$$\frac{\partial}{\partial x}K_n(x,t,u)\Big|_{u=u_1,t=0}=-nu_1K_{n+m}(x,t,u_1).$$

Theorem 3. *Let $\{K_n(x,t,u)\}$ satisfy 1^0-5^0 and let f be a function p-times continuously differentiable on $[0,\infty)$. Then*

$$L_n(f,x)=\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{n^2\psi_n(0)}\right)K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}$$

is infinitely differentiable on $[0,\gamma_n]$ and

$$L_n^{(p)}(f,x)=\sum_{\nu=0}^{\infty}\Delta_{h^{-1}}^p f\left(\frac{\nu}{h}\right)\frac{n(n+m)\ldots(n+(p-1)m)}{\nu!}x^\nu\times$$

$$\times K_{n+(\nu+pm)}(x,0,\alpha_n\psi_n(0))(-\alpha_n\psi_n(0))^{\nu+p},$$

where $\Delta_{h^{-1}}$ denotes the difference operator with the step $h^{-1}=\left(n^2\psi_n(0)\right)^{-1}$ and $\Delta_{h^{-1}}^p$ is the p-th iterate of this operator.

Proof. The series is absolutely convergent on $[0,\gamma_n]$ because for each $f\in C_\rho$, we have

$$\sum_{\nu=0}^{\infty}\left|f\frac{\nu}{n^2\psi_n(0)}\right|K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}\le M_f\sum_{\nu=0}^{\infty}\left(1+\frac{\nu}{n^2\psi_n(0)}\right)^2\times$$

$$\times K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}=M_f\left(1+\left(\frac{\alpha_n}{n}\right)^2\frac{n+m}{n}x^2+\right.$$

$$\left.+\left(\frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}+2\frac{\alpha_n}{n}\right)x\right)\le M_f\left(1+\left(\frac{\alpha_n}{n}\right)^2\frac{n+m}{n}\gamma_n^2+\left(\frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}2\frac{\alpha_n}{n}\right)\gamma_n\right).$$

By ν -multiple application of property 4^0, we obtain

$$K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))=(-1)^\nu x^\nu n(n+m)\ldots(n+(\nu-1)m)K_{n+\nu m}(x,0,\alpha_n\psi_n(0))$$

or applying property 5^0, we get

$$\frac{d}{dx}\left(K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\right)=(-1)^\nu\nu x^{\nu-1}n(n+m)\ldots(n+(\nu-1)m)K_{n+\nu m}(x,0,\alpha_n\psi_n(0))-$$

$$-(-1)^\nu x^\nu n(n+m)\ldots(n+\nu m)\alpha_n\psi_n(0)K_{n+(\nu+1)m}(x,0,\alpha_n\psi_n(0)).$$

Hence

$$\frac{d}{dx}L_n(f;x)=\sum_{\nu=0}^{\infty}f\left(\frac{\nu}{n^2\psi_n(0)}\right)\frac{d}{dx}\left(K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\right)\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}=$$

$$=\sum_{\nu=0}^{\infty}\left[f\left(\frac{\nu+1}{n^2\psi_n(0)}\right)-f\left(\frac{\nu}{n^2\psi_n(0)}\right)\right]x^\nu\,\frac{n(n+m)\ldots(n+\nu m)}{\nu!}\times \tag{10}$$

$$\times K_{n+(\nu+1)m}(x,0,\alpha_n\psi_n(0))(\alpha_n\psi_n(0))^{\nu+1}.$$

By property 5^0, $K_n(x,t,u)$ is bounded on the interval $[0,\gamma_n]$ for each fixed n. Thus, we can write

$$K_n(x,0,\alpha_n\psi_n(0))\le\max_{0\le x\le\gamma_n}K_n(x,0,\alpha_n\psi_n(0))=K_n(x_n^*,0,\alpha_n\psi_n(0)).$$

Since $|f(x)|\le M_f(1+x^2)$,

$$\left|f\left(\frac{\nu+1}{n^2\psi_n(0)}\right)-f\left(\frac{\nu}{n^2\psi_n(0)}\right)\right|\le 2M_f\left(1+\left(\frac{\nu+1}{n^2\psi_n(0)}\right)^2\right)\le 6M_f\left(1+\left(\frac{\nu}{n^2\psi_n(0)}\right)^2\right).$$

On the other hand, we have

$$K_{n+m}^{(\nu)}(x,0,\alpha_n\psi_n(0))=(-x)^\nu(n+m)(n+2m)\ldots(n+\nu m)\times$$
$$\times K_{n+(\nu+1)m}(x,0,\alpha_n\psi_n(0)).$$

Using this equality in (10) we get

$$\frac{d}{dx}L_n(f,x)=n\alpha_n\psi_n(0)\sum_{\nu=0}^{\infty}\left[f\left(\frac{\nu+1}{n^2\psi_n(0)}\right)-f\left(\frac{\nu}{n^2\psi_n(0)}\right)\right]\times$$
$$\times K_{n+m}^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}.$$

The upper bound of this series is

$$A=6nM_f\frac{\alpha_n\psi_n(0)}{(n^2\psi_n(0))^2}\sum_{\nu=0}^{\infty}\left(1+\left(\frac{\nu}{n^2\psi_n(0)}\right)^2\right)\times$$
$$\times K_{n+m}^{(\nu)}(x_n^*,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}$$

and we can write

$$A=6M_f\frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}\left[1+\left(\frac{\alpha_n}{n}\right)^2\frac{n+m}{n}(x_n^*)^2+\frac{1}{n^2\psi_n(0)}x_n^*\right].$$

Consequently majorant series of the series (10) is convergent which implies that the series (10) is uniformly convergent on $[0,\gamma_n]$.

Now, we can establish the existence of p-th derivatives of $L_n(f;x)$ for certain class of functions f. Since the series is uniformly convergent on $[0,\gamma_n]$, repeating the process of differentiation which can justified in the same way, we reach the result

$$L_n^{(p)}(f;x)=\sum_{\nu=0}^{\infty}\Delta_{h^{-1}}^{p}f\left(\frac{\nu}{h}\right)\frac{n(n+m)\ldots(n+(\nu+p-1)m)}{\nu!}x^\nu\times$$
$$\times K_{n+(\nu+p)m}(x,0,\alpha_n\psi_n(0))(\alpha_n\psi_n(0))^{\nu+p}. \tag{11}$$

Now, considering property 4^0, we can write

$$K_{n+pm}^{(\nu)}(x,0,\alpha_n\psi_n(0))=(n+pm)(n+(p+1)m)\ldots(n+(p+\nu-1)m)\times$$

$$\times(-x)^\nu K_{n+(\nu+p)m}(x,0,\alpha_n\psi_n(0)).$$

From this equality, (11) can be written in the form of

$$L_n^{(p)}(f;x)=\sum_{\nu=0}^{\infty}\Delta_{h^{-1}}^{p}f\left(\frac{\nu}{h}\right)\frac{n(n+m)\ldots(n+(p-1)m)}{\nu!}x^\nu\times$$
$$\times K_{n+(\nu+p)m}^{(\nu)}(x,0,\alpha_n\psi_n(0))(-\alpha_n\psi_n(0))^\nu(\alpha_n\psi_n(0))^p.$$

It is well-known that

$$\Delta^p f\left(\frac{\nu}{n^2\psi_n(0)}\right)=p!\frac{1}{\left(n^2\psi_n(0)\right)^p}\left[f;\frac{\nu}{n^2\psi_n(0)},\frac{\nu+1}{n^2\psi_n(0)},\ldots,\frac{\nu+p}{n^2\psi_n(0)}\right],$$

where $\left[f;\frac{\nu}{n^2\psi_n(0)},\frac{\nu+1}{n^2\psi_n(0)},\ldots,\frac{\nu+p}{n^2\psi_n(0)}\right]$ is the divided difference of f at the points $\frac{\nu}{n^2\psi_n(0)},\frac{\nu+1}{n^2\psi_n(0)},\ldots,\frac{\nu+p}{n^2\psi_n(0)}$.

Moreover, since the divided difference of f satisfies the equality

$$\left[f;\frac{\nu}{n^2\psi_n(0)},\frac{\nu+1}{n^2\psi_n(0)},\ldots,\frac{\nu+p}{n^2\psi_n(0)}\right]=\frac{1}{p!}f^{(p)}(\zeta),\quad \frac{\nu}{n^2\psi_n(0)}<\zeta<\frac{\nu+p}{n^2\psi_n(0)}.$$

Taking $\zeta=\frac{\nu+\theta_\nu p}{n^2\psi_n(0)}$, $0<\theta_\nu<1$, we have

$$L_n^{(p)}(f;x)=\left(\frac{\alpha_n}{n}\right)^p\frac{n(n+m)\ldots(n+(p-1)m)}{n^p}\sum_{\nu=0}^{\infty}f^{(p)}\left(\frac{\nu+\theta_\nu p}{n^2\psi_n(0)}\right)\times$$
$$\times K_{n+pm}^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}. \tag{12}$$

Theorem 4. *If the function $f(x)$ is $(p-1)$- times continuously differentiable on $[0,\infty)$, and its p-th derivative $f^{(p)}$ belongs to the class $Lip_M\alpha$, $0<\alpha\le 1$, then*

$$\lim_{n\to\infty}\sup_{x[0,\gamma_n]}\frac{\left|L_n^{(p)}(f,x)-f^{(p)}(x)\right|}{1+x^\alpha}=0.$$

Proof. Denoting $\left(\frac{\alpha_n}{n}\right)^p\frac{n(n+m)\ldots(n+(p-1)m)}{n^p}$ by $I_{n,p}$ we can see that $\lim_{n\to\infty}I_{n,p}=1$. Considering the series (12), we can write the inequality

$$\left|L_n^p(f;x)-f^{(p)}(x)\right|\le I_{n,p}\sum_{\nu=0}^{\infty}\left|f^{(p)}\left(\frac{\nu+\theta_\nu p}{n^2\psi_n(0)}\right)-f^p(x)\right|P_{\nu,n}(x)+\left|f^{(p)}(x)\right|\left|I_{n,p}-1\right|,$$

where

$$P_{\nu,n}(x)=K_{n+pm}^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}.$$

Since $f\in Lip_M\alpha$ we have

$$\left|L_n^p(f;x)-f^{(p)}(x)\right|\le I_{n,p}M\sum_{\nu=0}^{\infty}\left|\frac{\nu+\theta_\nu p}{n^2\psi_n(0)}-x\right|^\alpha P_{\nu,n}(x)+\left|f^{(p)}(x)\right|\left|I_{n,p}-1\right|,$$

where M is Lipschitz constant.

Applying the Hölder's inequality, we obtain

$$\left|L_n^{(p)}(f,x)-f^{(p)}(x)\right|\le I_{n,p}M\left(\sum_{\nu=0}^{\infty}\left(\frac{\nu+p}{n^2\psi_n(0)}-x\right)^2P_{\nu,n}(x)\right)^{\alpha/2}+\left|f^{(p)}(x)\right|\left|I_{n,p}-1\right|. \quad (13)$$

From the inequality

$$\left|f^{(p)}(t)-f^{(p)}(x)\right|\le M|t-x|^{\alpha} \quad \text{for all } t,x\ge 0$$

we can write

$$\left|f^{(p)}(x)\right|\le\left|f^{(p)}(0)\right|+Mx^{\alpha}\le\max\left(\left|f^{(p)}(0)\right|,M\right)\left(1+x^{\alpha}\right)=M_f\left(1+x^{\alpha}\right), \quad (14)$$

where $M_f=\max\left(\left|f^{(p)}(0)\right|,M\right)$.

Considering the equalities (4) and by simple calculations, we obtain

$$\sum_{\nu=0}^{\infty}\left(\frac{\nu+\theta_\nu p}{n^2\psi_n(0)}-x\right)^2P_{\nu,n}(x)=\left(\left(\frac{\alpha_n}{n}-1\right)^2-2\frac{pm}{n}\frac{\alpha_n}{n}+\frac{mn+2pm(n+pm)}{n^2}\left(\frac{\alpha_n}{n}\right)\right)x^2+$$

$$+\left(\left(1+\frac{3pm}{n}+2p\right)\frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}-2p\frac{1}{n^2\psi_n(0)}\right)x+\frac{p^2}{\left(n^2\psi_n(0)\right)^2}. \quad (15)$$

If we substitute the expressions (14) and (15) in inequality (13) we have

$$\left|L_n^{(p)}(f,x)-f^{(p)}(x)\right|\le I_{n,p}M\left\{ax^2+bx+c\right\}^{\alpha/2}+M_f\left(1+x^{\alpha}\right)\left|I_{n,p}-1\right|, \quad (16)$$

where a,b,c denote first, second and third terms in right hand side of (15), respectively.

Therefore, we reach the result

$$\lim_{n\to\infty}\sup_{x\in[0,\gamma_n]}\frac{\left|L_n^{(p)}(f,x)-f^{(p)}(x)\right|}{1+x^{\alpha}}=0.$$

Now, we give a rate of convergence of $L_n^{(p)}(f,x)$ to $f^{(p)}$, when $f^{(p)}\in Lip_M\alpha$.

Theorem 5. *Let f be a $(p-1)$-times continuously differentiable function and let $f^{(p)}\in Lip_M\alpha$. Then*

$$\sup_{x\in[0,\gamma_n]}\frac{\left|L_n^{(p)}(f,x)-f^{(p)}(x)\right|}{1+x^{\alpha}}=O\left(\left[\max\left\{\frac{1}{n},\frac{1}{n^2\psi_n(0)}\right\}\right]^{\alpha/2}\right)$$

for a sufficiently large n.

Proof. By condition (3) we get

$$I_{n,p}=1+O\left(\frac{1}{\left(n^2\psi_n(0)\right)^p}\right).$$

Using this equality and the condition (3) in the inequality (16), we can write

$$\left|L_n^{(p)}(f,x)-f^{(p)}(x)\right|=M\left(O\left(\frac{1}{\left(n^2\psi_n(0)\right)^2}\right)+O\left(\frac{1}{n}\right)O\left(\frac{1}{n^2\psi_n(0)}\right)+O\left(\frac{1}{n}\right)\right)x^2+$$

$$+O\left(O\left(\frac{1}{n^2\psi_n(0)}\right)+O\left(\frac{1}{n}\right)O\left(\frac{1}{n^2\psi_n(0)}\right)\right)x+O\left(\frac{1}{n^2\psi_n(0)}\right)\Bigg\}^{\alpha/2}\left(1+O\left(\frac{1}{\left(n^2\psi_n(0)\right)^p}\right)\right)+$$

$$+M_f\left(1+x^{\alpha}\right)O\left(\frac{1}{\left(n^2\psi_n(0)\right)^p}\right).$$

Consequently, we obtain

$$\sup_{x\in[0,\gamma_n]}\frac{\left|L_n^{(p)}(f,x)-f^{(p)}(x)\right|}{1+x^\alpha}=\left\{O\left(\frac{1}{\left(n^2\psi_n(0)\right)^2}\right)+O\left(\frac{1}{n}\right)O\left(\frac{1}{n^2\psi_n(0)}\right)+O\left(\frac{1}{n^2\psi_n(0)}\right)\right\}^{\alpha/2}$$

$$\times\left(1+O\left(\frac{1}{\left(n^2\psi_n(0)\right)^p}\right)\right)+O\left(\frac{1}{\left(n^2\psi_n(0)\right)^p}\right)=\left\{O\left(\frac{1}{n^2\psi_n(0)}\right)+O\left(\frac{1}{n}\right)O\left(\frac{1}{n^2\psi_n(0)}\right)\right\}^{\alpha/2}\times$$

$$\times\left(1+O\left(\frac{1}{\left(n^2\psi_n(0)\right)^p}\right)\right)+O\left(\frac{1}{\left(n^2\psi_n(0)\right)^p}\right)=O\left(\left[\max\left\{\frac{1}{n},\frac{1}{n^2\psi_n(0)}\right\}\right]^{\alpha/2}\right)$$

for a sufficiently large n.

Corollary. *If the function f satisfies the Lipschitz condition on $[0,\infty)$ then*

$$\sup_{x\in[0,\gamma_n]}\frac{\left|L_n(f,x)-f(x)\right|}{1+x^\alpha}=O\left(\left[\max\left\{\frac{1}{n},\frac{1}{n^2\psi_n(0)}\right\}\right]^{\alpha/2}\right)$$

for a large n.

In conclusion we consider an application of our results to the special case

$$K_n(x,t,u)=\left[1-\frac{ux}{1+t}\right]^n,\quad n=1,2,\ldots.$$

It seen that this function satisfies to need conditions with $m=-1$, $\gamma_n=\frac{1}{n^2\psi_n(0)}$ and in this case operators (1) has the from

$$L_n(f;x)=\sum_{\nu=0}^{n}f\left(\frac{\nu}{n^2\psi_n(0)}\right)\binom{n}{\nu}(\alpha_n\psi_n(0)x)^\nu\left[x\alpha_n\psi_n(0)\right]^{n-\nu}. \qquad (17)$$

By this reason our Theorems 1-5 holds also for operators (17) too. Moreover, choosing in (17)

$$\alpha_n=n,\quad \psi_n(0)=\frac{1}{nb_n},$$

where $\lim_{n\to\infty}b_n=\infty$, $\lim_{n\to\infty}\frac{b_n}{n}=0$, we obtain $\gamma_n=b_n$ and (17) gives a classical Bernstein-Chlodowsky polynomials

$$B_n(f;x)=\sum_{\nu=0}^{n}f\left(\frac{\nu b_n}{n}\right)\binom{n}{\nu}\left(\frac{x}{b_n}\right)^\nu\left(1-\frac{x}{b_n}\right)^{n-\nu}.$$

Corresponding statement on weighted approximation by polynomials $B_n(f;x)$ may be obtained from the Theorems 1-5 above.

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Gadjiev A.D.
Institute of Mathematics and Mechanics of AS Azerbaijan.
9, F. Agayeva str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20.

Ispir N.
Ankara University, Science of Faculty, Department of Mathematics.
Ankara, Turkey.

Received September 2, 1999; Revised Mart 22, 2000.
Translated by authors.