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ON ONE GENERALIZED BALLISTIC PROBLEM IN HILBERT SPACE

Abstract

In this work the problem on existence and absence of solution of the problems of one generalized ballistic problem for the non-linear abstract differential equation of the second orde which has linear general part is studied.

Let us consider in Hilbert space H the differential equation

$$\ddot{u}(t)+Au(t)=f(t,u(t)),\ t\in(0,T], \tag{1}$$

where A is the positive determined self-conjugated operator in H, $f(t,u)$ is the given function.

It's desired to find such a couple $(u(t),\tau)$ that $\tau\in(0,T]$, and $u(t)\in C^{(2)}([0,T];H)$ and satisfies the equation (1) in $[0,T]$ and the following supplementary conditions

$$u(0)=a, \tag{2}$$

$$u(\tau)=b \tag{3}$$

under the condition

$$\int_0^\tau \ddot{u}^2(t)dt=\vartheta, \tag{4}$$

where a, b are the given elements from space H, $\vartheta>0$ is the given number.

Let's denote the energetic space of operator A as H_A.

For the first time the problem of (1)-(4) type was formulated and investigated by Shamilov A.Kh.

In the suggested work the existence and absence of the solution of the problem (1)-(4) are established.

First let's consider the following auxiliary corresponding to (1) linear non-homogeneous equation

$$\ddot{u}(t)+Au(t)=\varphi(t),\ t\in[0,T] \tag{5}$$

with conditions (2)-(4).

Theorem 1. *Let*

1) *A be the self-conjugated positive determined operator acting in Hilbert space H and having the discrete spectrum;*
2) *$a,b\in H_A$; $a\neq b$;*
3) *$\varphi(t)\in C([0,T];H)$;*
4) *for some ξ, $\xi\in(0,T)$ it takes place the inequality*

$$\left[\|b-a\|_{H_A} Cth\sqrt{\lambda_1}\xi+\|b\|_{H_A}+\sqrt{T}\left(\int_0^T\|\varphi(s)\|^2ds\right)^{1/2}\right]^2<\vartheta,$$

where λ_1 is the least eigen number of operator A.

Then the problem (5), (2)-(4), where the conditions (2), (3) are fulfilled even in sense of the norm of space H_A, has though one solution $(u(t),\tau):\tau\in(0,T]$ $u(t)\in C^{(2)}([0,\tau];H)$. If instead of condition (4) the inequality

$$\left[\left|b_n - a_n\right|\sqrt{\lambda_n}\frac{1}{Sh\sqrt{\lambda_n}T} - \|b\|_{H_A} - \sqrt{T}\left(\int_0^T\|\varphi(s)\|^2 ds\right)^{1/2}\right]^2 > \vartheta \qquad (6)$$

is fulfilled, where a_n, b_n are Fourier n-th coefficient of elements a, b by the system of eigen functions $\{u_n\}$ of operator A, and λ_n is the corresponding n-th eigen number of operator A, then the problem (5), (2)-(4) has not the solution $(u(t),\tau): u(t) \in C^2([0,\tau]; H)$ $\tau \in (0,T]$.

Theorem 1 is proved by analogy to [3].

Remark. Solution (5), (2), (3) under the conditions of Theorem 1 by virtue of [2] can be given by the formula

$$u(t) = \sum_{n=1}^{\infty}\left(a_n\frac{Sh\sqrt{\lambda_n}(\tau - t)}{Sh\sqrt{\lambda_n}\tau} + b_n\frac{Sh\sqrt{\lambda_n}t}{Sh\sqrt{\lambda_n}\tau} - \int_0^{\tau}G_n(t,s,\tau)\varphi_n(s)ds\right)u_n, \qquad (7)$$

where

$$G_n(t,s,\tau) = \frac{1}{\sqrt{\lambda_n}Sh\sqrt{\lambda_n}\tau}\begin{cases} Sh\sqrt{\lambda_n}t Sh\sqrt{\lambda_n}(\tau - s) & 0 \le t \le s \\ Sh\sqrt{\lambda_n}(\tau - t)Sh\sqrt{\lambda_n}S & s \le t \le \tau \end{cases} \qquad (8)$$

a_n, b_n, $\varphi_n(t)$ are Fourier coefficients according to a, b and $\varphi(t)$ by the system $\{u_n\}$, $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$, $\lambda_n \to \infty$ are eigen numbers of operator A.

Further, with help of the obtained results for the considered auxiliary problem, the problem (1)-(4) is reduced to the system of operator equations and the method of dependent variables suggested by Shamilov A.Kh. [2] is used.

Let's introduce the following operator

$$(P_0\varphi)(t) = \sum_{n=1}^{\infty}\left(a_n\frac{Sh\sqrt{\lambda_n}(\tau - t)}{Sh\sqrt{\lambda_n}\tau} + b_n\frac{Sh\sqrt{\lambda_n}t}{Sh\sqrt{\lambda_n}\tau} - \int_0^{\tau}G_n(t,S,\tau)\varphi_n(s)ds\right)u_n, \qquad (9)$$

where $G_n(t,s,\tau)$ is determined by (8).

Let's consider the operator

$$(P_\tau\varphi)(t) \stackrel{def}{=} f(t,(P_0\varphi)(t)). \qquad (10)$$

Using (9) and (10), we reduce the problem (1)-(3) to the following operator equation

$$\varphi(t) = (P_\tau\varphi)(t). \qquad (11)$$

Taking (9) into account in (4) we obtain

$$g_\tau\varphi \stackrel{def}{=} \int_0^{\tau}\left(\frac{d(P_0\varphi)(t)}{dt}\right)^2 dt = \vartheta. \qquad (12)$$

So we obtain the following system of operator equations equivalent to the problem

$$\begin{cases} \varphi(t) = (P_\tau\varphi)(t) \\ g_\tau\varphi = \vartheta \end{cases} \qquad (13)$$

Theorem 2. *Let*

1) *A is the self-conjugated positive determined operator acting in Hilbert space H and having the discrete spectrum;*
2) *$a,b \in H_A$; $a \ne b$;*

3) *$f:[0,T]\times H\to H$ is continuous and satisfies the Lipshits condition by the norm $C([0,t]:H)$, i.e. $\forall\bar{u}(t),u(t)\in C([0,T]:H)$, $\forall(t,\bar{u}(t))$, $(t,u(t))\in[0,T]\times H$, $\|f(t,\bar{u}(t))-f(t,u(t))\|_c\le\theta\|\bar{u}(t)-u(t)\|_c$;*
4) *for some $\xi\in(0,T]$ the inequality*
$$\sqrt{\vartheta}-\|b\|_{H_A}-\|b-a\|_{H_A}Cth\sqrt{\lambda_1}\xi>0;$$
5) *the inequalities are fulfilled*
$$\theta_0=\frac{\theta T}{2\sqrt{\lambda_1}}<1,\quad \frac{\|f(t,(P_0O)(t))\|}{1-\theta_0}\le\frac{\sqrt{\vartheta}-\|b\|_{H_A}-\|b-a\|_{H_A}Cth\sqrt{\lambda_1}\xi}{\xi}$$
$\forall\tau\in(0,T]$, *where* $(P_0O)(t)=\sum_{n=1}^{\infty}\left(a_n\frac{Sh\sqrt{\lambda_n}(\tau-t)}{Sh\sqrt{\lambda_n}\tau}+b_n\frac{Sh\sqrt{\lambda_n}t}{Sh\sqrt{\lambda_n}\tau}\right)u_n$.

Then the problem (1)-(4), where the conditions (2) and (3) are fulfilled even in sense of the norm of space H_A, has though one solution $(u(t);\tau):u(t)\in C^{(2)}([0,\tau]:H)$; $\tau\in(0,T]$.

Proof. It is known that the problem (1)-(4) is equivalent to the system (13). In order to solve the system (10) first we find the immovable point of operator $P_\tau\varphi$.

It is known the problem (5), (2)-(4) has a solution for $\varphi(x)\in C([0,T]:H)$ by Theorem 1. Also by this theorem $(P_0\varphi)(t)\in C^{(2)}([0,T]:H)$. So $(P_\tau\varphi)(t)=f(t)$, $((P_0\varphi)(t))$ also belongs to $C([0,T]:H)$ by the condition (3) of this theorem. So we obtain
$$P_\tau:C([0,T]:H)\to C([0,T]:H).$$
The problem (5), (2)-(4) is also solvable by virtue of condition (4) of Theorem 1 for the following $\varphi(t)$: $\varphi(t)\in C([0,T]:H)$
$$\|\varphi\|_C\le\frac{\sqrt{\vartheta}-\|b\|_{H_A}-\|b-a\|_{H_A}Cth\sqrt{\lambda_1}\xi}{\xi}.$$

Now let's prove that operator P_τ is the compressive mapping. We take
$$\forall\varphi_1(t),\varphi_2(t)\in\left\{\varphi\in C[0,T]:H\,\middle|\,\|\varphi\|_C\le R\right\}\overset{def}{=}S_R,$$
where R is some given positive number.
$$(P_\tau\varphi_1)(t)-(P_\tau\varphi2)(t)=f(t,(P_0\varphi_1)(t))-f(t,(P_0\varphi_2)(t)).$$
Further, by virtue of Lipshits condition
$$\|(P_\tau\varphi_1)(t)-(P_\tau\varphi_2)(t)\|_C\le\theta\|(P_0\varphi_1)(t)-(P_0\varphi_2)(t)\|_C.$$
In the last inequality using (9) we obtain
$$(P_0\varphi_1)(t)-(P_0\varphi_2)(t)=\sum_{n=1}^{\infty}\left(-\int_0^\tau G_n(t,s,\tau)(\varphi_{n,1}(s)-\varphi_{n,2}(s))ds\right)u_n.$$

Hence using Bunyakovsky inequality we obtain
$$\|(P_0\varphi_1)(t)-(P_0\varphi_2)(t)\|^2=\sum_{n=1}^{\infty}\left(-\int_0^\tau G_n(t,s,\tau)(\varphi_{n,1}(s)-\varphi_{n,2}(s))ds\right)^2\le$$
$$\le\sum_{n=1}^{\infty}\int_0^\tau G_n^2(t,s\tau)ds\int_0^\tau(\varphi_{n,1}(s)-\varphi_{n,2}(s))^2ds.$$

Bu virtue of [2-3] it is valid the inequality

$$\left|G_n(t,s,\tau)\right| \le \frac{1}{2\sqrt{\lambda_n}}. \tag{14}$$

Then

$$\left\|(P_0\varphi_1)(t)-(P_0\varphi_2)(t)\right\|^2 = \sum_{n=1}^{\infty}\int_0^{\tau}\left(\frac{1}{2\sqrt{\lambda_n}}\right)^2 ds\int_0^{\tau}\left(\varphi_{n,1}(s)-\varphi_{n,2}(s)\right)^2 ds \le$$

$$\le \frac{\tau}{4\lambda_1}\sum_{n=1}^{\infty}\int_0^{\tau}\left(\varphi_{n,1}(s)-\varphi_{n,2}(s)\right)^2 ds = \frac{\tau}{4\lambda_1}\int_0^{\tau}\sum_{n=1}^{\infty}\left(\varphi_{n,1}(s)-\varphi_{n,2}(s)\right)^2 ds =$$

$$= \frac{\tau}{4\lambda_1}\int_0^{\tau}\left\|\varphi_1(s)-\varphi_2(s)\right\|_H^2 ds \le \frac{\tau}{4\lambda_1}\cdot\tau\max_{[0,\tau]}\left\|\varphi_1(s)-\varphi_2(s)\right\|_H^2 = \frac{\tau^2}{4\lambda_1}\left\|\varphi_1(s)-\varphi_2(s)\right\|_{C([0,\tau]H)}^2.$$

Hence

$$\left\|(P_0\varphi_1)(t)-(P_0\varphi_2)(t)\right\|_C \le \frac{\tau}{2\sqrt{\lambda_1}}\left\|\varphi_1(s)-\varphi_2(s)\right\|_C. \tag{15}$$

By virtue of (15) from (13) we obtain

$$\left\|(P_\tau\varphi_1)(t)-(P_\tau\varphi_2)(t)\right\|_C \le \theta\frac{\tau}{2\sqrt{\lambda_1}}\left\|\varphi_1(s)-\varphi_2(s)\right\|_C \le \frac{\theta T}{2\sqrt{\lambda_1}}\left\|\varphi_1(s)-\varphi_2(s)\right\|_C. \tag{16}$$

From (16) it follows that operator P_τ is the compressive mapping in $C([0,T]:H)$ if $\theta_0 = \frac{\theta T}{2\sqrt{\lambda_1}} < 1$.

Let

$$R = \frac{\sqrt{\vartheta} - \|b\|_{H_A} - \|b-a\|_{H_A} Cth\sqrt{\lambda_1}\xi}{\xi}. \tag{17$_0$}$$

Then from (5) of this theorem we obtain

$$\frac{\|P_\tau O\|_C}{1-\theta_0} \le R.$$

Hence

$$\|P_\tau O\|_{C([0,T]H)} \le R(1-\theta_0). \tag{17}$$

The last inequality shows that the compressing mapping P_τ represents the sphere S_R with radius R into itself. By virtue of compressing mappings principle P_τ in sphere S_R has the only immovable point $\varphi(t)$. Moreover, the coefficient of compression doesn't depend on τ. Consequently, the immovable point φ continuously depends on τ on any closed segment $0 < t_0 \le \tau \le T$.

Let take into account the found immovable point $\varphi(t;\tau)$ of the operator in the second equation of system (13). Then it takes the form:

$$g_\tau\varphi(t\tau) = \int_0^{\tau}\dot{u}^2(t)dt = \left\|\dot{u}(t)\right\|_{L_2([0,\tau])}^2 =$$

$$= \left\|\sum_{n=1}^{\infty}\left(-a_n\sqrt{\lambda_n}\frac{Ch\sqrt{\lambda_n}(\tau-t)}{Sh\sqrt{\lambda_n}\tau} + b_n\sqrt{\lambda_n}\frac{Ch\sqrt{\lambda_n}t}{Sh\sqrt{\lambda_n}\tau} - \int_0^{\tau}\frac{\partial G_n(t,s,\tau)}{\partial t}\varphi_n(s;\tau)ds\right)u_n\right\|_{L_2}^2 = \vartheta. \tag{9'}$$

Now the whole problem has been reduced to the establishment of the sufficient conditions of existence or absence of the solution of the equation (9').

Let's study that $g_\tau \varphi$ by parameter τ. It's clear that $g_\tau \varphi$ can be represented in the following form:

$$g_\tau \varphi = \sum_{n=1}^{\infty}\left\{(b_n - a_n)\sqrt{\lambda_n}\frac{Ch\sqrt{\lambda_n}(\tau - t)}{Sh\sqrt{\lambda_n}\tau} + b_n\sqrt{\lambda_n}\frac{Ch\sqrt{\lambda_n}t - Ch\sqrt{\lambda_n}(\tau - t)}{Sh\sqrt{\lambda_n}\tau} - \right.$$

$$\left. - \int_0^\tau \frac{\partial G_n}{\partial t}\varphi_n(s)ds\right\}^2 \le \sum_{n=1}^{\infty} 3\left\{(b_n - a_n)^2\lambda_n\frac{Ch^2\sqrt{\lambda_n}(\tau - t)}{Sh^2\sqrt{\lambda_n}\tau} + \right.$$

$$\left. + b_n^2\lambda_n\frac{\left(Ch\sqrt{\lambda_n}t - Ch\sqrt{\lambda_n}(\tau - t)\right)^2}{Sh^2\sqrt{\lambda_n}\tau} + \left(\int_0^\tau \frac{\partial G_n(t,s,\tau)}{\partial t}\varphi_n(s)ds\right)^2\right\} = 3(u_1 + u_2 + u_3), \tag{18}$$

where

$$u_1 = \sum_{n=1}^{\infty}(b_n - a_n)^2\lambda_n\frac{Ch^2\sqrt{\lambda_n}(\tau - t)}{Sh^2\sqrt{\lambda_n}\tau};\quad u_2 = \sum b_n^2\lambda_n\frac{\left(Ch\sqrt{\lambda_n}t - Ch\sqrt{\lambda_n}(\tau - t)\right)^2}{Sh^2\sqrt{\lambda_n}\tau};$$

$$u_3 = \sum_{n=1}^{\infty}\left(\int_0^\tau \frac{\partial G_n(t,s,\tau)}{\partial t}\varphi_n(s)ds\right)^2. \tag{19}$$

As far as $0 < t \le \tau$

$$\frac{Ch(\tau - t)}{Sh\tau} \le \frac{Ch\tau}{Sh\tau} = Cth\tau\ ;\quad \frac{Cht}{Sh\tau} \le \frac{Ch\tau}{Sh\tau} = Cth\tau. \tag{20}$$

From (19)

$$u_1 = \sum_{n=1}^{\infty}(b_n - a_n)^2\lambda_n\frac{Ch^2\sqrt{\lambda_n}(\tau - t)}{Sh^2\sqrt{\lambda_n}\tau} \le \sum_{n=1}^{\infty}(b_n - a_n)^2\lambda_n Cth^2\sqrt{\lambda_n}\tau \le$$

$$\le Cth^2\sqrt{\lambda_1}\tau\sum_{n=1}^{\infty}(b_n - a_n)^2\lambda_n = Cth\sqrt{\lambda_1}\tau\|b - a\|_{H_a}^2. \tag{21}$$

From (19)

$$u_2 = \sum_{n=1}^{\infty} b_n^2\lambda_n\frac{\left(Ch\sqrt{\lambda_n}t - Ch\sqrt{\lambda_n}(\tau - t)\right)^2}{Sh^2\sqrt{\lambda_n}\tau} = \sum_{n=1}^{\infty} b_n^2\lambda_n\left(\frac{Sh\sqrt{\lambda_n}\left(t - \tau/2\right)}{Ch^2\sqrt{\lambda_n}\,\tau/2}\right)^2.$$

As far as $0 < t \le \tau$

$$\frac{Sh\left(t - \tau/2\right)}{Ch\,\tau/2} \le \frac{Sh\,\tau/2}{Ch\,\tau/2} = th\,\tau/2. \tag{22}$$

From the last one and the fact that $th\xi \le 1$ for $\xi \in (0,\infty)$ we obtain

$$u_2 \le \sum_{n=1}^{\infty} b_n^2\lambda_n th^2\sqrt{\lambda_n}\,\tau/2 \le \sum_{n=1}^{\infty} b_n^2\lambda_n = \|b\|_{H_A}^2. \tag{23}$$

From (19) using Bunyakovsky inequality by virtue of [2, 3]

$$\left|\frac{\partial G_n(t,s,\tau)}{\partial t}\right| \le 1 \tag{24}$$

we obtain

$$u_3 = \sum_{n=1}^{\infty}\left(\int_0^\tau \frac{\partial G_n(t,s,\tau)}{\partial t}\varphi_n(s)ds\right)^2 \le \sum_{n=1}^{\infty}\int_0^\tau\left(\frac{\partial G_n(t,s,\tau)}{\partial t}\right)^2 ds \cdot \int_0^\tau \varphi_n^2(s)ds \le$$

$$\leq \sum_{n=1}^{\infty} \tau \cdot \int_0^{\tau} \varphi_n^2(s)ds = \tau \int_0^{\tau} \sum_{n=1}^{\infty} \varphi_n^2(s)ds = \tau \int_0^{\tau} \|\varphi(s)\|_H^2 ds . \tag{25}$$

Taking (21), (23) and (25) into account in (18) we obtain the following

$$g_{\tau}\varphi \leq 3\left(\|b-a\|_{H_A}^2 Cth\sqrt{\lambda_1}\tau + \|b\|_{H_A}^2 + \tau \int_0^{\tau} \|\varphi(s)\|^2 ds \right). \tag{26}$$

From the last one it follows that the initial series in (18) uniformly converges. By virtue of its uniform convergence of the series it will be continuous by τ. Therefore $g_{\tau}\varphi$ is continuous by parameter $\tau \in (0,T]$.

Now let's consider the equation (9')

$$g_{\tau}\varphi = \vartheta$$

or

$$\left\| \sum_{n=1}^{\infty} \left\{ (b_n - a_n)\sqrt{\lambda_n} \frac{Ch\sqrt{\lambda_n}(\tau - t)}{Sh\sqrt{\lambda_n}\tau} + b_n\sqrt{\lambda_n} \frac{Ch\sqrt{\lambda_n}t - Ch\sqrt{\lambda_n}(\tau - t)}{Sh\sqrt{\lambda_n}\tau} - \right.\right.$$

$$\left.\left. - \int_0^{\tau} \frac{\partial G_n}{\partial t} \varphi_n(s)ds \right\} u_n \right\|_{L_2([0,\tau])}^2 = \vartheta .$$

Let's consider the left-hand side

$$g_{\tau}\varphi = \|V_1 + V_2 + V_3\|^2 , \tag{27}$$

$$V_1 = \sum_{n=1}^{\infty} (b_n - a_n)\sqrt{\lambda_n} \frac{Ch\sqrt{\lambda_n}(\tau - t)}{Sh\sqrt{\lambda_n}\tau} u_n ; \quad V_3 = \sum_{n=1}^{\infty} \left(- \int_0^{\tau} \frac{\partial G_n(t,s,\tau)}{\partial t} \varphi_n(s)ds \right) u_n ;$$

$$V_2 = \sum_{n=1}^{\infty} b_n \sqrt{\lambda_n} \frac{Ch\sqrt{\lambda_n}t - Ch\sqrt{\lambda_n}(\tau - t)}{Sh\sqrt{\lambda_n}\tau} u_n . \tag{28}$$

From (28) and taking (20) into account

$$\|V_1\|^2 = \sum_{n=1}^{\infty} (b_n - a_n)^2 \lambda_n \frac{Ch^2\sqrt{\lambda_n}(\tau - t)}{Sh^2\sqrt{\lambda_n}\tau} \leq \sum_{n=1}^{\infty} (b_n - a_n)^2 \lambda_n Cth^2\sqrt{\lambda_n}\tau \leq$$

$$\leq Cth^2\sqrt{\lambda_1}\tau \sum_{n=1}^{\infty} (b_n - a_n)^2 \lambda_n = Cth^2\sqrt{\lambda_1}\tau \|b - a\|_{H_A}^2 ,$$

$$\|V_1\| \leq \|b - a\|_{H_A} Cth\sqrt{\lambda_1}\tau . \tag{29}$$

From (28) and taking into account (22)

$$\|V_2\|^2 = \sum_{n=1}^{\infty} b_n^2 \lambda_n \left(\frac{Ch\sqrt{\lambda_n}t - Ch\sqrt{\lambda_n}(\tau - t)}{Sh\sqrt{\lambda_n}\tau} \right)^2 \leq \|b\|_{H_A}^2 ,$$

$$\|V_2\| \leq \|b\|_{H_A}^2 . \tag{30}$$

From (28) and taking into account (25)

$$\|V_3\|^2 = \sum_{n=1}^{\infty} \left(\int_0^{\tau} \frac{\partial G_n(t,s,\tau)}{\partial t} \varphi_n(s)ds \right)^2 \leq \tau \int_0^{\tau} \|\varphi(s)\|^2 ds ,$$

$$\|V_3\| \leq \sqrt{\tau} \left(\int_0^{\tau} \|\varphi(s)\|^2 ds \right)^{1/2} . \tag{31}$$

Taking into account (29), (30) and (31) in (27)

$$g_\tau\varphi = \|V_1 + V_2 + V_3\|^2 \le (\|V_1\| + \|V_2\| + \|V_3\|)^2 \le$$

$$\le \left(\|b-a\|_{H_A} Cth\sqrt{\lambda_1}\tau + \|b\|_{H_A} + \sqrt{\tau}\left(\int_0^\tau \|\varphi(s)\|^2 ds\right)^{1/2} \right)^2 .$$

Thus

$$g_\tau\varphi \le \left[\|b-a\|_{H_A} Cth\sqrt{\lambda_1}\tau + \|b\|_{H_A} + \sqrt{\tau}\left(\int_0^\tau \|\varphi(s)\|^2 ds\right)^{1/2} \right]^2 . \tag{32}$$

From (25)

$$\|V_3\|^2 \le \tau\int_0^\tau \|\varphi(s)\|^2 ds \le \tau \max_{[0,\tau]} \|\varphi(s)\|^2 \int_0^\tau ds = \tau^2 \|\varphi\|^2_{C([0,\tau]H)},$$

$$\|V_3\| \le \tau\|\varphi\|_C . \tag{31'}$$

Taking (31') into account in (32)

$$g_\tau\varphi \le \left[\|b-a\|_{H_A} Cth\sqrt{\lambda_1}\tau + \|b\|_{H_A} + \tau\|\varphi\|_{C([0,\tau]H)} \right]^2 . \tag{32'}$$

Let's now estimate $g_\tau\varphi$ below

$$g_\tau\varphi = \|V_1 + V_2 + V_3\|^2_{L_2([0,\tau])}.$$

From (28)

$$\|V_1\|^2 = \sum_{n=1}^\infty (b_n - a_n)^2 \lambda_n \frac{Ch^2\sqrt{\lambda_n}(\tau - t)}{Sh^2\sqrt{\lambda_n}\tau}.$$

So as by the known property of function chx the inequality $ch\,x \ge 1$, $x \in (0,\infty)$ has place

$$\|V_1\|^2 = \sum_{n=1}^\infty (b_n - a_n)^2 \lambda_n \frac{Ch^2\sqrt{\lambda_n}(\tau - t)}{Sh^2\sqrt{\lambda_n}\tau} \ge$$

$$\ge \sum_{n=1}^\infty (b_n - a_n)^2 \lambda_n \frac{1}{Sh^2\sqrt{\lambda_n}\tau} \ge (b_n - a_n)^2 \lambda_n \frac{1}{Sh^2\sqrt{\lambda_n}\tau}, \quad \forall n \in N$$

$$\|V_1\|^2 \ge (b_n - a_n)^2 \lambda_n \frac{1}{Sh^2\sqrt{\lambda_n}\tau}, \quad \forall n \in N$$

or

$$\|V_1\| \ge |b_n - a_n|\sqrt{\lambda_n}\frac{1}{Sh\sqrt{\lambda_n}\tau}, \quad \forall n \in N. \tag{33}$$

Hence it follows that if $b - a \ne 0$, then there is such n that $b_n - a_n \ne 0$ and the product $|b_n - a_n|\sqrt{\lambda_n}\frac{1}{Sh\sqrt{\lambda_n}\tau}$ for $\tau \to 0$ becomes rather big. So $\|V_1\| \to +\infty$ for $\tau \to 0$.

From (30) it follows that $\|V_2\|$ becomes restricted upper $\left(\|V_2\| \le \|b\|_{H_A}\right)$. And from (31) also it follows that $\|V_3\|$ becomes restricted upper too.

Taking (33), (30) and (31) into account in (27)

$$\|V_1 + V_2 + V_3\| \ge \|V_1\| + \|V_2\| + \|V_3\| \ge |b_n - a_n|\sqrt{\lambda_n}\frac{1}{Sh\sqrt{\lambda_n}\tau} - \|b\|_{H_A} - \sqrt{\tau}\left(\int_0^\tau \|\varphi(s)\|^2 ds\right)^{1/2},$$

$$g_\tau\varphi = \|V_1 + V_2 + V_3\|^2 \geq \left[|b_n - a_n|\sqrt{\lambda_n}\frac{1}{Sh\sqrt{\lambda_n}\tau} - \|b\|_{H_A} - \sqrt{\tau}\left(\int_0^\tau \|\varphi(s)\|^2 ds\right)^{1/2}\right]^2.$$

Thus,

$$g_\tau\varphi \geq \left[|b_n - a_n|\sqrt{\lambda_n}\frac{1}{Sh\sqrt{\lambda_n}\tau} - \|b\|_{H_A} - \sqrt{\tau}\left(\int_0^\tau \|\varphi(s)\|^2 ds\right)^{1/2}\right]^2. \qquad (34)$$

From the last one it follows that for $\tau \to 0$ the first component in the right-hand side of (34) tents to $+\infty$, and the second and third components are restricted. So we obtain that $g(\tau) \to +\infty$ for $\tau \to 0 (g(0) = +\infty)$. So for all $\vartheta > 0$ for $\tau \to 0$ $g(\tau) - \vartheta \to +\infty$.

Moreover, the inequality (32') is fulfilled

$$g_\tau\varphi \leq \left[\|b-a\|_{H_A} Cth\sqrt{\lambda_1}\tau + \|b\|_{H_A} + \tau\|\varphi\|_C\right]^2 \leq \left[\|b-a\|_{H_A} cth\sqrt{\lambda_1}\tau + \|b\|_{H_A} + \tau R\right]^2.$$

As we have proved that (32') is valid for all $\tau \in (0,T]$, so (32') is valid for $\xi \in (0,T]$ namely

$$g_\xi\varphi \leq \left[\|b-a\|_{H_A} Cth\sqrt{\lambda_1}\xi + \|b\|_{H_A} + \xi R\right]^2.$$

Taking (17) into account in the last inequality we obtain

$$g_\xi\varphi \leq \left[\|b-a\|_{H_A} Cth\sqrt{\lambda_1}\xi + \|b\|_{H_A} + \sqrt{\vartheta} - \|b\|_{H_A} - \|b-a\|_{H_A} Cth\sqrt{\lambda_1}\xi\right]^2 = \vartheta.$$

Hence

$$g_\xi\varphi - \vartheta < 0. \qquad (35)$$

Taking into account by virtue of continuity of $g_\tau\varphi$ on $(0,T]$ we obtain that equation $g_\tau\varphi = \vartheta$ has on the segment $(0,\xi]$ at least one solution.

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