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## THE REGULARITY TEST OF A BOUNDARY POINT FOR NON-UNIFORMLY DEGENERATING SECOND ORDER ELLIPTIC EQUATIONS

**Abstract**

*A class of second order divergent structure equations with non-uniform power degeneration is considered in the paper. The regularity test of the Wiener's type for a boundary point with respect to the first boundary value problem for such equations is proved.*

**Introduction.** Let $D$ be the bounded domain, arranged in $n$- dimensional Euclidean space $E_n$ of points $x=(x_1,\dots,x_n)$, $n\geq 3$, and $\partial D$ be its boundary, $O\in\partial D$.

Consider in $D$ the first boundary value problem

$$Lu=\sum_{i,j=1}^{n}\left(a_{ij}(x)u_j\right)_i=0,\quad x\in D;\quad u\big|_{\partial D}=\varphi, \tag{1}$$

where $\|a_{ij}(x)\|$ is a real symmetric matrix with elements measurable in $D$,

$$u_i=\frac{\partial u}{\partial x_i}\quad (i,j=1,2,\dots,n),\quad \varphi\in C(\partial D).$$

Assume that with respect to the coefficients of the operator $L$ it is fulfilled the condition

$$\mu\sum_{i=1}^{n}\lambda_i(x)\xi_i^2\leq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leq\mu^{-1}\sum_{i=1}^{n}\lambda_i(x)\xi_i^2, \tag{2}$$

where $\mu\in(0,1],\, x\in D,\, \xi\in E_n,\, \lambda_i(x)=\left(|x|_\alpha\right)^{\alpha_i}$, $|x|_\alpha=\sum_{i=1}^{n}|x_i|^{\frac{2}{2+\alpha_i}}$, $\alpha=(\alpha_1,\dots,\alpha_n)$,

$\alpha_i\in\left[0,\frac{2}{n-1}\right)$, $i=1,\dots,n$.

The goal of this article is to find the Wiener's type regularity test of the boundary point $O$ with respect to the problem (1). Note that for the Laplace equation the classical result in this direction was obtained by N. Wiener [1]. The Wiener's test was transferred to the equations with smooth coefficients in [2-3]. In [4] it was established that the Wiener's test is valid for arbitrary, uniformly elliptic second order equations of divergent structure with measurable coefficients. Elliptic equations with uniform degeneration were considered in [5]. In [6] the regularity test for a boundary point was obtained for elliptic equations with weak (so-called logarithmic) non-uniform degeneration. Note that none equation with non-uniform power degeneration satisfies the conditions of paper [6]. In the present paper the regularity test for a for a boundary point was obtained for class of second order divergent elliptic equations with non-uniform power degeneration. Concerning divergent structure elliptic equations we note the results obtained in the indicated direction in papers [7-11].

### $1^0$. Some notations, definitions and subsidiary statements.

Let $\Sigma$ be a sufficiently great radius closed ball with a center in the origin of coordinates, $\overline{D}\subset\Sigma$.

Denote by $W^1_{p,\Lambda}(D), \overset{\circ}{W}{}^1_{p,\Lambda}(D)$ the closure of functions correspondingly from $C^\infty(\overline{D})$, $C_0^\infty(D)$ at the following norm

$$\left(\int_D |u|^p dx + \int_D \sum_{i=1}^n \lambda_i(x)\left|\frac{\partial u}{\partial x_i}\right|^p dx\right)^{1/p}, \quad 1<p<\infty .$$

The space adjoint to $\overset{\circ}{W}{}^1_{p,\Lambda}(D)$ denote by $W^{-1}_{p',\Lambda}(D)$:

$$W^{-1}_{p',\Lambda}(D)=\left\{T=f_0+\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}: f_0\in L_{p'}(D), f_i\in L_{p',\lambda_i^{-1}}(D), i=1,\ldots,n\right\},$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

The function $u\in W^1_{2,\Lambda}(D)$ we call the solution of the equation (1), if it satisfies the following integral identity

$$\int_D \sum_{i,j=1}^n a_{ij}(x)u_i v_j dx=0 \quad \text{for any} \quad v\in \overset{\circ}{W}{}^1_{2,\Lambda}(D).$$

If

$$\int_D \sum_{i,j=1}^n a_{ij}(x)u_i v_j dx\le 0 \quad \text{for any} \quad v\in \overset{\circ}{W}{}^1_{2,\Lambda}(D), v\ge 0 ,$$

then the function $u\in W^1_{2,\Lambda}(D)$ is called a subsolution of the equation (1). If the function $-u(x)\in W^1_{2,\Lambda}(D)$ is the subsolution of the equation (1), then $u(x)$ is called a supersolution. We say that the charge $\nu$ belongs to $W^{-1}_{p',\Lambda}$, if for any $\varphi\in C_0^\infty(D)$

$$\left|\int_D \varphi d\nu\right|\le c\|\varphi\|_{W^1_{p,\Lambda}(D)}.$$

Further, we shall denote positive constants by $c$. For $k>0$, $R>0$, $x^0\in E_n$ by $\mathcal{E}_R^{x^0}(k)$ we denote the ellipsoid

$$\left\{x:\sum_{i=1}^n \frac{(x_i-x_i^0)^2}{R^{\alpha_i}}\le (kR)^2\right\}.$$

Let $u\in W^1_{p,\Lambda}(D)$. We say that $u\ge a$ on $E\subset\overline{D}$ in the sense of $W^1_{p,\Lambda}(D)$, if there exists a sequence of functions $\{\varphi_k\}\in Lip(\overline{D})$ such that $\varphi_k(x)\ge a, x\in E$ and $\varphi_k\to u$, $(k\to\infty)$ by the norm of $W^1_{p,\Lambda}(D)$. Let $u(x)$ be a measurable function. The function $u^{(\varepsilon)}(x)=\min\{u(x),\varepsilon\}$ is called $\varepsilon$-truncation of $u(x)$. Let

$$D(u,v)=\int_\Sigma \sum_{i,j=1}^n a_{ij}(x)u_i v_j dx .$$

**Lemma 1.** *Let* $p_0>0$ *is a sufficiently large number,* $p>p_0, T\in W^{-1}_{p,\Lambda}(\Sigma)$ *and* $Lu=T$ *in the sense of* $W^1_{2,\Lambda}(D)$, $u\in \overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma)$. *Then, the function* $u(x)$ *is continuous by Hölder in* $\overline{\Sigma}$ *and*

$$\max_{\Sigma}|u| \le c_1 \|T\|_{W^{-1}_{p,\Lambda}(\Sigma)},$$

$$\max_{\substack{x,y\in\Sigma \\ |x-y|\le\rho}} |u(x)-u(y)| \le c_2 \rho^{\alpha} \|T\|_{W^{-1}_{p,\Lambda}(\Sigma)}.$$

**Lemma 2.** *Let* $u \in \overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma)$, $\varepsilon > 0$. *Then, if* $u^{(\varepsilon)}$ *is a* $\varepsilon$*- truncation of the function* $u(x)$, *then* $u^{(\varepsilon)} \in \overset{\circ}{W}{}^1_{2,\Lambda}(D)$. *Moreover, if* $\{\varphi_j\} \in C_0^{\infty}(D)$ *and* $\varphi_j \to u$, $(j \to \infty)$ *by the norm* $\overset{\circ}{W}{}^1_{2,\Lambda}(D)$, *then* $\varphi_j^{(\varepsilon)} \to u^{(\varepsilon)}$, $(j \to \infty)$ *weakly in* $\overset{\circ}{W}{}^1_{2,\Lambda}(D)$. *Moreover*

$$\|u^{(\varepsilon)}\|_{W^1_{2,\Lambda}(D)} \le \|u\|_{W^1_{2,\Lambda}(D)}.$$

These lemmas are proved by a standard method as for instance in [5].

**Lemma 3.** *If* $u \in W^1_{2,\Lambda}(\Sigma)$ *is the subsolution of the equation (1), non-positive on* $\partial D$ *in the sense of* $W^1_{2,\Lambda}(D)$, *then the function* $u(x)$ *is non-positive almost everywhere on* $D$.

**Proof.** Let $\varepsilon > 0$ be an arbitrary number. Then, $u - u^{(\varepsilon)} \ge 0$ in $D$ and by Lemma 2 $u - u^{(\varepsilon)} \in \overset{\circ}{W}{}^1_{2,\Lambda}(D)$. Since $u$ is a subsolution of the equation (1), then

$$\int_D \sum_{i,j=1}^{n} a_{ij} u_i \varphi_j dx \le 0$$

for any function $\varphi \in \overset{\circ}{W}{}^1_{2,\Lambda}(D)$, $\varphi \ge 0$.

Assume $\varphi = u - u^{(\varepsilon)}$. We have

$$\int_D \sum_{i,j=1}^{n} a_{ij} u_i \left(u - u^{(\varepsilon)}\right)_j dx \le 0.$$

Since

$$\int_D \sum_{i,j=1}^{n} a_{ij} u_i^{(\varepsilon)} \left(u - u^{(\varepsilon)}\right)_j dx = 0,$$

then

$$\int_D \sum_{i,j=1}^{n} a_{ij} \left(u - u^{(\varepsilon)}\right)_i \left(u - u^{(\varepsilon)}\right)_j dx \le 0.$$

By the Friedrichs type inequality [12] and (2) we get that $u(x) = u^{(\varepsilon)}(x)$ almost everywhere in $D$. Since $\varepsilon$ is an arbitrary positive number, then $u \le 0$ almost everywhere in $D$. The Lemma is proved.

Denote

$$V_{\Sigma}(\mathcal{K}) = \left\{u \in \overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma) : u \ge 1 \text{ on } \mathcal{K} \text{ in the sense of } W^1_{2,\Lambda}(\mathcal{K})\right\},$$

where $\mathcal{K} \subset \Sigma$ is some compact set. If $\Sigma = E_n$, then we denote $V_{E_n}(\mathcal{K})$ by $V(\mathcal{K})$.

## $2^0$. Capacity and capacitary potential.

The number $cap_{\Sigma}(\mathcal{K}) = \inf_{u \in V_{\Sigma}(\mathcal{K})} \int_{\Sigma} \sum_{i,j=1}^{n} a_{ij}(x) u_i u_j dx$ is called capacity of the compactum $\mathcal{K}$ with respect to the ball $\Sigma$, generated by the operator $L$.

The number $cap(\mathcal{K})=\inf\limits_{u\in V(\mathcal{K})}\int\limits_{E_n}\sum\limits_{i,j=1}^{n}a_{ij}(x)u_iu_jdx$ is called capacity of the compactum $\mathcal{K}$, generated by the operator $L$.

**Lemma 4.** *There exists a unique function $u\in V_\Sigma(\mathcal{K})$, for which $cap_\Sigma(\mathcal{K})=D(u,u)$. Moreover, $u=1$ on $\mathcal{K}$ in the sense of $W^1_{2,\Lambda}(\Sigma)$ and $D(u,v)\geq 0$ for any function $v\in \overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma)$ such that $v\geq 0$ on $\mathcal{K}$ in the sense of $W^1_{2,\Lambda}(\Sigma)$.*

**Proof.** It is easy to show that $V_\Sigma(\mathcal{K})$ is a convex and closed set in $\overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma)$. Then by the known theorem on a functional analysis there exists a unique function $u\in V_\Sigma(\mathcal{K})$, possessing minimal norm among the elements of $V_\Sigma(\mathcal{K})$. By [12] a bilinear form of $D(u,v)$ is a scalar product in $\overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma)$, therefore $cap_\Sigma(\mathcal{K})=D(u,u)$. By Lemma 2 we get $u=1$ on $\mathcal{K}$ in the sense of $W^1_{2,\Lambda}(\Sigma)$.

Let $v\in \overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma)$, $v\geq 0$ on $\mathcal{K}$ in the sense of $W^1_{2,\Lambda}(\Sigma)$ and $\varepsilon>0$. It is obvious that $u+\varepsilon v\in V_\Sigma(\mathcal{K})$ for any $\varepsilon>0$. Then

$$D(u+\varepsilon v,u+\varepsilon v)\geq D(u,u).$$

Hence it follows that

$$2\varepsilon D(u,v)+\varepsilon^2D(v,v)\geq 0.$$

Therefore $D(u,v)\geq 0$. The Lemma is proved.

The function $u$ is called a capacitary potential of the compactum $\mathcal{K}$.

**Corollary 2.** *Capacitary potential of the compactum $\mathcal{K}$ is the supersolution of the equation (1) in $\Sigma$.*

**Lemma 5.** *Let $u$ be a capacitary potential of some compactum $\mathcal{K}\subset\Sigma$. Then $Lu=0$ in $\Sigma\backslash\mathcal{K}$ and $0\leq u(x)\leq 1$ almost everywhere in $\Sigma$.*

**Proof.** Let $u$ be a capacitary potential of some compactum $\mathcal{K}\subset\Sigma$. Then $Lu=0$ in the sense of $W^1_{2,\Lambda}(\Sigma\backslash\mathcal{K})$. Indeed, let $v(x)$ be any function from $\overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma\backslash\mathcal{K})$. Denote

$$\widetilde{v}(x)=\begin{cases}v(x), & x\in\Sigma\backslash\mathcal{K}\\ 0, & x\in\mathcal{K}\end{cases}$$

It is obvious that $\widetilde{v}\in \overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma)$. Then by Lemma 4

$$\int\limits_{\Sigma\backslash\mathcal{K}}\sum_{i,j=1}^{n}a_{ij}(x)u_iv_jdx=D(u,\widetilde{v})\geq 0.$$

Since $v$ any function from $\overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma\backslash\mathcal{K})$, then $Lu=0$ in $\Sigma\backslash\mathcal{K}$. On $\partial\Sigma$ $u=0$ in the sense of $W^1_{2,\Lambda}(\Sigma)$. On $\partial\mathcal{K}$ $u=1$ in the sense of $W^1_{2,\Lambda}(\Sigma)$, i.e. on $\partial(\Sigma\backslash\mathcal{K})$ $0\leq u(x)\leq 1$. On the other hand, $u=1$ on $\mathcal{K}$ in the sense of $W^1_{2,\Lambda}(\Sigma)$, i.e. $u=1$ almost everywhere on $\mathcal{K}$. Therefore $0\leq u(x)\leq 1$ almost everywhere in $\Sigma$. The Lemma is proved.

Let $\varphi \in C_0^\infty(\Sigma)$, $\varphi(x) \geq 0$, $x \in \mathcal{K}$. By Lemma 4 $D(u,\varphi) \geq 0$, where $u$ is a capacitary. Then by Schwartz's theorem there exists a unique measure $\mu$, such that $D(u,\varphi) = \int\limits_\Sigma \varphi d\mu$.

**Corollary 3.** *$S(\mu) \subset \mathcal{K}$, where $S(\mu)$ is the support of measure $\mu$.*

Measure $\mu$ is called the capacity distribution of the compactum $\mathcal{K}$.

**Lemma 6.** *Let $\mu$ be the capacitary distribution of the compactum $\mathcal{K} \subset \Sigma$. Then $S(\mu) \subset \partial\mathcal{K}$ and $\mu(\mathcal{K}) = cap_\Sigma(\mathcal{K})$.*

**Proof.** Let $u$ be a capacitary potential of $\mathcal{K}$. By Lemma 4 there exists a sequence of functions $\{\varphi_j\} \in C_0^\infty(\Sigma)$, $\varphi_j(x) = 1$, $x \in \mathcal{K}$, $j = 1,2,\ldots$, $\varphi_j \to u$, $(j \to \infty)$ by the norm of $W_{2,\Lambda}^1(\Sigma)$. Let $\psi$ be an arbitrary function from $C_0^\infty(\Sigma)$ such that $S(\psi) \subset \mathcal{K}^0$ ($\mathcal{K}^0$ is the interior part of $\mathcal{K}$). We have

$$\int\limits_\Sigma \psi d\mu = D(u,\psi) = \lim_{j\to\infty} D(\varphi_j, \psi)$$

and

$$cap_\Sigma(\mathcal{K}) = D(u,u) = \lim_{j\to\infty} D(u,\varphi_j) = \lim_{j\to\infty} \int\limits_{\partial\mathcal{K}} \varphi_j d\mu = \mu(\mathcal{K}).$$

The Lemma is proved.

**Corollary 4.** *The capacitary distribution of the compactum $\mathcal{K} \subset \Sigma$ belongs $W_{2,\Lambda}^{-1}(\Sigma)$.*

Let $T \in W_{p,\Lambda}^{-1}(\Sigma)$, $p \geq p_0$, where a positive number $p_0$ is chosen by Lemma 1. By Theorem 1 [12] there exists a unique function $u(x) \in \mathring{W}_{2,\Lambda}^1(\Sigma)$, for which $Lu = T$ in the sense of $W_{2,\Lambda}^1(\Sigma)$. Let $G(T) = u$. Then by Lemma 1 $G$ maps $W_{p,\Lambda}^{-1}(\Sigma)$ into $C(\Sigma)$ and it is a linear bounded operator. Denote by $M(\Sigma)$ a class of finite charges in $\Sigma$.

## $3^0$. Weak solutions.

**Definition.** *Let $\mu$ be the charges of bounded variation on $\Sigma$. We say that the function $u \in L_1(\Sigma)$ is a weak solution of the equation $Lu = \mu$, equal to zero on the boundary $\partial\Sigma$, if it satisfies the equality*

$$\int\limits_\Sigma u L\varphi dx = \int\limits_\Sigma \varphi d\mu$$

*for any $\varphi \in \mathring{W}_{2,\Lambda}^1(\Sigma) \cap C(\overline{\Sigma})$, such that $Lu \in C(\overline{\Sigma})$.*

**Definition.** *The function $u \in L_1(\Sigma)$ is called a weak solution of the equation $Lu = \mu$ $(\mu \in M(\Sigma))$, converging to zero in $\partial\Sigma$, if*

$$\int\limits_\Sigma u(x)\psi(x)dx = \int\limits_\Sigma G(\psi)d\mu$$

*for any $\psi \in C(\overline{\Sigma})$. It is obvious that if $\psi \in C(\overline{\Sigma})$, then $\psi \in L_p(\Sigma)$ for any $p \geq 1$ and $\psi \in W_{p\Lambda}^{-1}(\Sigma)$. Therefore $\int\limits_\Sigma G(\psi)d\mu$ has sense for any $\psi \in C(\overline{\Sigma})$.*

**Lemma 7.** *Let $\mu \in M(\Sigma)$. Then there exists a unique weak solution of the equation $Lu = \mu$. Moreover $u \in \overset{\circ}{W}{}^1_{p',\Lambda}(\Sigma)$,*

$$\|u\|_{W^1_{p',\Lambda}(\Sigma)} \le c_3 \|\mu\|_{M(\Sigma)}, \quad \text{where} \quad 1 \le p' \le p_0$$

$\frac{1}{p'} + \frac{1}{p_0} = 1$, *and $p_0$ is a constant from Lemma 1.*

The following statements we cite without proof.

**Lemma 8.** *If the charge $\mu \in M(\Sigma)$ is the measure, then a weak solution of the equation $Lu = -\mu$ is a non-negative function almost everywhere in $\Sigma$.*

**Lemma 9.** *Assume that $\mu \in M(\Sigma)$ is the measure and $\mu \in W^{-1}_{2,\Lambda}(\Sigma)$. Then a weak solution of the equation $Lu = -\mu$ belongs to $\overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma)$. Moreover $Lu = -\mu$ in the sense of $\overset{\circ}{W}{}^1_{2,\Lambda}(\Sigma)$.*

**Lemma 10.** *Let $B_1 = \mathcal{E}^y_r(1)$, $B_2 = \mathcal{E}^y_r(2)$ and $Lu = 0$ in the sense of $W^1_{2,\Lambda}(B_2)$. Then it is valid the inequality*

$$\int_{B_1} \sum_{i=1}^n \lambda_i(x) u_i^2 dx \le c_4 \frac{1}{r^2} \int_{B_2} u^2 dx. \qquad (3)$$

### 4⁰. Green's function and its properties.

Fix $y \in \Sigma$. Denote by $g(x,y)$ a weak solution of the equation $Lg = -\delta_y$, where $\delta_y$ is a Dirac's measure, concentrated at the point $y$. $g(x,y)$ is called a Green's function of the sphere $\Sigma$. It possesses a number of properties of classical Green's function.

**Lemma 11.** $g(x,y) \in W^1_{2,\Lambda}(\Sigma \setminus \mathcal{E}^y_r(1))$ *for any $r > 0$. Moreover, we can so change the function $g(x,y)$ on the set of Lebesgue's zero measure that the obtained function will be continuous by Hölder in $\Sigma \setminus \{y\}$ and converge to zero on $\partial\Sigma$ (see [13]).*

**Lemma 12.** *For any $\mu \in M(\Sigma)$ the integral*

$$u(x) = \int_\Sigma g(x,y) d\mu(y)$$

*exists almost everywhere in $\Sigma$, moreover $u(x)$ is a weak solution of the equation $Lu = -\mu$.*

**Lemma 13.** *Let $\psi \in C(\overline{\Sigma})$. Then*

$$G(\psi)(y) = \int_\Sigma g(x,y)\psi(x) dx$$

*is the solution of the equation $L\varphi = \psi$.*

**Proof.** We have

$$\int_\Sigma g(x,y)\psi(x)dx = < g(\cdot,y), \psi > = < G^*(\delta_y), \psi > = < \delta_y, G(\psi) > = G(\psi)(y).$$

The Lemma is proved.

**Lemma 14.** $g(x,y) = g(y,x)$ *for all $x, y \in \Sigma \times \Sigma$.*

**Lemma 15.** *Let $\mathcal{E}^x_r(2) \subset \Sigma$, $y \in \partial\mathcal{E}^x_r(1)$. Then*

$$\frac{c_5}{cap_\Sigma\left(\mathcal{E}_r^x(1)\right)} \le g(x,y) \le \frac{c_6}{cap_\Sigma\left(\mathcal{E}_r^x(1)\right)}.$$

**Proof.** Let $\mu$ is a capacitary distribution of $\mathcal{E}_r^x(1)$, and

$$u(z) = \int_\Sigma g(z,\tau)d\mu(\tau)$$

is a capacitary potential of $\mathcal{E}_r^x(1)$.

According to previously proved $S(\mu) \subset \partial\mathcal{E}_r^x(1)$. Therefore

$$u(z) = \int_{\partial\mathcal{E}_r^x(1)} g(z,\tau)d\mu(\tau).$$

Since $x \notin \partial\mathcal{E}_r^x(1)$, then $x \ne \tau$, therefore $g(x,\tau)$ is a continuous function and $u(z)$ is continuous at the point $x$, so

$$1 = u(x) = \int_{\partial\mathcal{E}_r^x(1)} g(x,\tau)d\mu(\tau).$$

Hence we get

$$\min_{\tau\in\partial\mathcal{E}_r^x(1)} g(x,\tau)\cdot cap_\Sigma\left(\mathcal{E}_r^x(1)\right) \le 1 \le \max_{\tau\in\partial\mathcal{E}_r^x(1)} g(x,\tau)\cdot cap_\Sigma\left(\mathcal{E}_r^x(1)\right).$$

With regard to Lemmas 8, 10 and the Harnack's inequality [14] we conclude

$$\max_{\tau\in\partial\mathcal{E}_r^x(1)} g(x,\tau) \le c_7 \min_{\tau\in\partial\mathcal{E}_r^x(1)} g(x,\tau).$$

The Lemma is proved.

**Lemma 16.** *Let* $x^0 \in \mathcal{E}_r^0(4)$. *Then*

$$c_8 r^{n-2}\prod_{i=1}^n r^{\alpha_i/2} \le cap\left(\mathcal{E}_r^{x^0}(1)\right) \le c_9 r^{n-2}\prod_{i=1}^n r^{\alpha_i/2}.$$

**Proof.** Let $\prod_r^{x^0} = \{x : |x_i - x_i^0| < r^{1+\alpha_i/2}, i = 1,\dots,n$ . Then $\mathcal{E}_r^{x^0}(1) \subset \prod_r^{x^0}$, therefore

$$cap\left(\mathcal{E}_r^{x^0}(1)\right) \le cap\left(\prod_r^{x^0}\right).$$

Consider the functions $f_i(t)$. $f_i(t) = 1, |t| < r^{1+\alpha_i/2}, f_i(t) = 0, |t| \ge 2r^{1+\alpha_i/2}, 0 \le f_i(t) \le 1$, $f_i(t) \in C_0^\infty(E_1)$, $i = 1,\dots,n$. We may assume that

$$\left|\frac{df_i}{dt}\right| \le \frac{c}{r^{1+\alpha_i/2}}.$$

Let $u(x) = \prod_{j=1}^n f_j\left(x_j - x_j^0\right)$. Then $u(x) = 1$ in $\prod_r^{x^0}$, $u(x) \in C_0^\infty(E_n)$. Besides it, $u(x) = 0$ outside $\widetilde{\prod}_r^{x^0} = \{x : |x_i - x_i^0| \le 2r^{1+\alpha_i/2}, i = 1,\dots,n$ , $\quad |u_i| \le \frac{c}{r^{1+\alpha_i/2}}$.

Let $x \in \widetilde{\prod}_r^{x^0}$. Then $|x|_\alpha = \sum_{j=1}^n |x_j|^{\frac{2}{2+\alpha_j}} \le \sum_{j=1}^n \left(|x_j - x_j^0| + |x_j^0|\right)^{\frac{2}{2+\alpha_j}}$.

But $|x_j - x_j^0| \le 2r^{1+\alpha_j/2}$, $|x_j^0| \le (4r)^{1+\alpha_j/2} \le 4^{1+\frac{\alpha^+}{2}}\cdot r^{1+\frac{\alpha_j}{2}} \le c_{10}r^{1+\alpha_j/2}$, $\lambda_j(x) \le c_{11}r^{\alpha_j}$, $j = 1,\dots,n$. Here $\alpha^+ = \max\{\alpha_1,\dots,\alpha_n\}$. Therefore $cap\left(\prod_r^{x_0}\right) \le \int_{E_n} a_{ij}u_iu_j dx \le$

$\leq \mu^{-1}\sum_{i=1}^{n}\int_{\widetilde{\Pi}_r^{x^0}}\lambda_i(x)u_i^2dx \leq c_{12}(\mu,n,\alpha)\times \quad \times r^{-2} mes\,\widetilde{\Pi}_r^{x_0} = c_{13}(\mu,n,\alpha)r^{n-2}\prod_{i=1}^{n} r^{\alpha_i/2}$. Denote by $Cap$ the capacity generated by the operator

$$L_0 = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\lambda_i(x)\frac{\partial}{\partial x_i}\right),$$

It is clear that $\mu Cap(E) \leq cap(E) \leq \mu^{-1}Cap(E)$.

Let $E$ be a compactum, $\widetilde{E}$ be it's image for transformation $y_i = k^{1+\alpha_i/2}\cdot x_i$, $i = 1,\ldots,n$, $k > 0$. Then

$$Cap(E) = \frac{Cap(\widetilde{E})}{k^{n-2}\prod_{i=1}^{n} k^{\alpha_i/2}}.$$

In particular $Cap(\Pi_r^{x^0}) = r^{n-2}\prod_{i=1}^{n} r^{\alpha_i/2} Cap(\Pi_1^{y^0})$.

Estimate $Cap(\Pi_1^{y^0})$. Let $u(x)$ is a capacitary potential of $\Pi_1^{y^0}$, and $\mu$ is a capacitary distribution of $\Pi_1^{y^0}$.

It is clear, that $S(\mu) \subset \partial\Pi_1^{y^0}$. We have

$$u(z) = \int_{\partial\Pi_1^{y^0}} g(z,\tau)d\mu(\tau),$$

since $y^0 \notin \partial\Pi_1^{y^0}$, then $u(z)$ is continuous at the point $y^0$,

$$1 = u(y^0) = \int_{\partial\Pi_1^{y^0}} g(y^0,\tau)d\mu(\tau).$$

For $\tau \in \partial\Pi_1^{y^0}$, $dist(y^0,\tau) \geq 1$. Therefore $g(y^0,\tau) \leq a(n,\alpha)$

$$1 \leq a\mu(\partial\Pi_1^{y^0}) = aCap(\Pi_1^{y^0})$$

and we get

$$Cap(\Pi_r^{x^0}) \geq ar^{n-2}\prod_{i=1}^{n} r^{\alpha_i/2}.$$

On the other hand

$$Cap(\Pi_r^{x^0}) \geq \mu Cap(\Pi_r^{x^0}) \geq \mu a r^{n-2}\prod_{i=1}^{n} r^{\alpha_i/2}.$$

Now after considering the inclusion $\mathcal{E}_r^{x^0}(1) \supset \Pi^{x^0}_{\frac{r}{n^{\frac{1}{2+\alpha^+}}}}$ Lemma is proved.

**Lemma 17.** *Let $\Sigma = Q_R^0$. Then, if $r \leq r^0$, $x^0 \in \mathcal{E}_r^0(4)$, $R$ is sufficiently large, then*

$$c_{14}r^{n-2}\prod_{i=1}^{n} r^{\alpha_i/2} \leq cap_\Sigma\left(\mathcal{E}_r^{x^0}(1)\right) \leq c_{15}r^{n-2}\prod_{i=1}^{n} r^{\alpha_i/2}.$$

**Lemma 18.** *Let $y \in \Sigma$. Then*

$$cap_\Sigma\{y\} = \lim_{r\to 0} cap_\Sigma\left(\mathcal{E}_r^{y}(1)\right) = 0.$$

## $5^0$. Generalized solution of Dirichlet's problem

**Lemma 19.** *Let the measure* $\mu \in M(\Sigma)$. *Then a weak solution* $u(x)$ *of the equation* $Lu = -\mu$ *is a lower semicontinuous function, i.e. for any* $x^0 \in \Sigma$

$$\varliminf_{x\to x^0} u(x) \ge u(x^0).$$

**Proof.** Let $g(x,y)$ is the Green's function. Then the weak solution of the $Lu = -\mu$ is represented in the form

$$u(x) = \int_\Sigma g(x,y)d\mu(y).$$

Fix $x^0 \in \Sigma$. Any measure $\mu \in M(\Sigma)$ is represented in the form of $\mu = \mu_1 + \mu_2$, where $\mu_1$ is an absolute continuous, and $\mu_2$ is a singular constituent, $\mu_1\{x^0\} = 0$, $\mu_2\{x^0\} = \mu\{x^0\}\delta_{x^0}$. Therefore

$$u(x) = \int_\Sigma g(x,y)d\mu_1(y) + \mu\{x^0\}g(x,x^0),$$

$$\varliminf_{x\to x^0} u(x) \ge \varliminf_{x\to x^0} \int_\Sigma g(x,y)d\mu_1(y) + \mu\{x^0\}g(x^0,x^0). \tag{4}$$

If $\mu\{x^0\} > 0$, then $\mu\{x^0\}g(x^0,x^0) = \infty$, then the Lemma is proved.

Let $\mu\{x^0\} = 0$. Choose the following sequence of functions

$$\varphi_k \in Lip(E_1), \quad \varphi_k \le \varphi_{k+1}$$

$\varphi_k = 0$ at the neighborhood of $0$, $\varphi_k = 1$ for $t \ge \frac{1}{k}$

$$\lim_{k\to\infty} \varphi_k(t) = \begin{cases} 0, & t = 0 \\ 1, & t \ne 0 \end{cases}$$

Let $g_k(x,y) = g(x,y)\varphi_k(|x-y|)$. It is obvious that $g_k(x,y) \le g_{k+1}(x,y)$, $\lim_{k\to\infty} g_k(x,y) =$ $= g(x,y)$, except the point $x = y$ and

$$\lim_{k\to\infty} \int_\Sigma g_k(x^0,y)d\mu_1(y) = \int_\Sigma g(x^0,y)d\mu_1(y). \tag{5}$$

But $\int_\Sigma g_k(x,y)d\mu_1(y)$ is a continuous function. Therefore

$$\int_\Sigma g_k(x^0,y)d\mu_1(y) = \lim_{x\to x^0} \int_\Sigma g_k(x,y)d\mu_1(y) = \varliminf_{x\to x^0} \int_\Sigma g_k(x,y)d\mu_1(y) \le$$
$$\le \varliminf_{x\to x^0} \int_\Sigma g(x,y)d\mu_1(y),$$

i.e. we obtained

$$\int_\Sigma g_k(x^0,y)d\mu_1(y) \le \varliminf_{x\to x^0} \int_\Sigma g(x,y)d\mu_1(y).$$

By using (5), we have

$$\int_\Sigma g(x^0,y)d\mu_1(y) \le \varliminf_{x\to x^0} \int_\Sigma g(x,y)d\mu_1(y).$$

Thus

$$u(x_0) = \int_\Sigma g(x^0,y)d\mu_1(y) \le \varliminf_{x\to x^0} \int_\Sigma g(x,y)d\mu_1(y) \le \varliminf_{x\to x^0} u(x)$$

and the Lemma is proved.

**Definition.** *The number* $cap^*_\Sigma(E)=\inf\{cap_\Sigma(U)\}$, *where the greatest lower bound is taken over all open sets containing* $E$ *is called the upper capacity of the set* $E$.

**Lemma 20.** *If* $\mu$ *is the measure,* $\mu\in W^{-1}_{2,\Lambda}$ *and for Berellian set* $E$

$$cap^*_\Sigma(E)=0, \quad then \quad \mu(E)=0.$$

**Corollary 5.** $\lim\limits_{x\to y} g(x,y)=\infty$.

*By* $H$ *denote the factor space* $W^1_{2,\Lambda}(D)\big/\overset{\circ}{W}{}^1_{2,\Lambda}(D)$. *Let the mapping* $B:H\to W^1_{2,\Lambda}(D)$ *be such, that if* $u=B\varphi$, *then* $Lu=0$ *in the sense of* $W^1_{2,\Lambda}(D)$ *and* $u-\overline{\varphi}\in\overset{\circ}{W}{}^1_{2,\Lambda}(D)$, *where* $\overline{\varphi}$ *is the representative of a class of equivalence* $\varphi$. *If the function* $\varphi$ *is bounded on* $\partial D$ *in the sense of* $W^1_{2,\Lambda}(D)$, *then by Lemma 3*

$$\sup_D|u|\le\max_{\partial D}|\varphi|, \tag{6}$$

*where by* $\max\limits_{\partial D}|\varphi|$ *we denote the greatest lower bound of numbers* $c$ *such that* $\overline{\varphi}\le c$, $-\overline{\varphi}\le c$. *By using (6) and (3), it is easy to deduce that*

$$|||B\varphi|||\le c_{16}\max_{\partial D}|\varphi|,$$

*where* $|||g|||=\sup\limits_{\overline{D}'\subset D}\delta\left(\sum\limits_{i=1}^{n}\int\limits_{D'}\lambda_i(x)g_i^2dx\right)^{1/2}+\max\limits_{D}|g|$ *and* $\delta$ *is the distance between* $\overline{D'}$ *and* $\partial D$. *Therefore,* $B$ *is a linear mapping of the subset* $B^{1,2}$ *of functions from* $H$, *bounded on* $\partial D$ *(in the sense of* $W^1_{2,\Lambda}(D)$*), in a space of functions with finite norm* $|||u|||$.

Since any continuous function $\varphi$ on $\partial D$ may be approximated in the norm $\max\limits_{\partial D}|\varphi|$ by the functions being smooth on any set containing $\overline{D}$, then the set $B^{1,2}$ is dense in the space of continuous on $\partial D$ functions $\varphi$ with the norm $\max\limits_{\partial D}|\varphi|$.

### $6^0$. Boundary point regularity.

**Definition.** *The point* $y\in\partial D$ *is called regular, if for any continuous on* $\partial D$ *functions* $\varphi(x)$ *for a generalized solution* $u=B\varphi$ *it is valid the equality*

$$\lim_{x\to y}u(x)=\varphi(y). \tag{7}$$

If there exist at least one continuous function $\varphi$ on $D$, for which (7) is not fulfilled, the point $y$ is called irregular.

**Lemma 21.** *The point* $y\in\partial D$ *is regular if and only if there exists the barrier* $\vartheta_y$ *in it.*

**Lemma 22.** *Let* $u(x)$ *is a capacitary potential of the compactum* $\mathcal{K}\subset\Sigma$. *Then*

$$u(y)=\varliminf_{\substack{x\to y\\ x\in\Sigma\setminus\mathcal{K}}}u(x)$$

**Corollary 6.** *Let $\mathcal{K} \subset \Sigma$ be some compactum $y \in \partial\mathcal{K}$, $u(x)$ is a capacitary potential of $\mathcal{K}$. For $u(y)=1$, it is necessary and sufficient that $u(x)$ was continuous at the point $y$.*

**Lemma 23.** *$y \in \partial D$ will be a regular boundary point if and only if for any $\rho > 0$ $u_\rho(y)=1$, where $u_\rho$ is a capacitary potential of the set $A_\rho = (\Sigma \setminus D) \cap \mathcal{E}_\rho^y(1)$.*

**Lemma 24.** *The point $y \in \partial D$ will be irregular if and only if*

$$\lim_{\rho \to 0} u_\rho(y) = 0. \tag{8}$$

**Lemma 25.** *If $\rho > r$, then*

$$\mu_r(A_r) = \mu_\rho(A_r) + \int_{A_\rho \setminus A_r} u_r d\mu_\rho . \tag{9}$$

**Proof.** It follows from Lemma 20, Corollary 4, and Fubini' s theorem that

$$\mu_r(A_r) = \int_{A_r} u_\rho d\mu_r = \int_{A_r} \left( \int_{A_\rho} g(x,y) d\mu_\rho(y) \right) d\mu_r(x) = \int_{A_\rho} u_r(y) d\mu_\rho(y) =$$

$$= \int_{A_r} u_r d\mu_\rho + \int_{A_\rho \setminus A_r} u_r d\mu_\rho = \mu_\rho(A_r) + \int_{A_\rho \setminus A_r} u_r d\mu_\rho .$$

The Lemma is proved.

**Corollary 7.** $\mu_\rho(A_r) \le \mu_r(A_r) = cap_\Sigma(A_r)$.

**Theorem 1.** For the point $O \in \partial D$ to be regular, it is necessary and sufficient that

$$\sum_{k=0}^{\infty} \frac{cap_\Sigma(A_{2^{-k}})}{cap_\Sigma(\mathcal{E}_{2^{-k}}^0(1))} = \infty . \tag{10}$$

**Proof. Necessity.** Let the condition (10) be not fulfilled, i.e.

$$\sum_{k=0}^{\infty} \frac{cap_\Sigma(A_{2^{-k}})}{cap_\Sigma(\mathcal{E}_{2^{-k}}^0(1))} < \infty . \tag{11}$$

Fix arbitrary $\varepsilon > 0$. Then it follows from (11) that there exists $m = m(\varepsilon)$ such that

$$\sum_{k=m}^{\infty} \frac{cap_\Sigma(A_{2^{-k}})}{cap_\Sigma(\mathcal{E}_{2^{-k}}^0(1))} < \varepsilon . \tag{12}$$

We have

$$u_{2^{-m}}(0) = \int_{A_{2^{-m}}} g(x,0) d\mu_{2^{-m}}(x),$$

where $\mu_{2^{-m}}$ is the capacitary distribution of $A_{2^{-m}}$. Further

$$\int_{A_{2^{-m}}} g(x,0) d\mu_{2^{-m}}(x) = \sum_{k=m}^{\infty} \int_{A_{2^{-k}} \setminus A_{2^{-k-1}}} g(x,0) d\mu_{2^{-m}}(x) \le \sum_{k=m}^{\infty} \sup_{x \in A_{2^{-k}} \setminus A_{2^{-k-1}}} g(x,0) \mu_{2^{-m}}(A_{2^{-k}}).$$

Thus

$$u_{2^{-m}}(0) \le \sum_{k=m}^{\infty} \sup_{x \in A_{2^{-k}} \setminus A_{2^{-k-1}}} g(x,0) cap_\Sigma(A_{2^{-k}}). \tag{13}$$

By the Harnack's type inequality and Lemma 15 we get

$$u_{2^{-m}}(0) \le c_{16} \sum_{k=m}^{\infty} \frac{cap_\Sigma(A_{2^{-k}})}{cap_\Sigma(\mathcal{E}_{2^{-k}}^0(1))} < c_{16} \cdot \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, then $\lim_{m\to\infty} u_{2^{-m}}(0) = 0$. By the Lemma 24 the point 0 is irregular.

**Sufficiency.** Let 0 be an irregular point. We have

$$u_{2^{-m}}(0) \geq \sum_{k=m}^{\infty} \inf_{x\in A_{2^{-k}}\setminus A_{2^{-k-1}}} g(x,0)\left[\mu_{2^{-m}}\left(A_{2^{-k}}\right) - \mu_{2^{-m}}\left(A_{2^{-k-1}}\right)\right]. \tag{14}$$

Further

$$\inf_{x\in A_{2^{-k}}\setminus A_{2^{-k-1}}} g(x,0) \geq c_{17} \frac{1}{cap_{\Sigma}\left(\mathcal{E}_{2^{-k}}^{0}(1)\right)}.$$

Using this in (14), we get

$$u_{2^{-m}}(0) \geq c_{17} \sum_{k=m}^{\infty} \frac{1}{cap_{\Sigma}\left(\mathcal{E}_{2^{-k}}^{0}(1)\right)}\left[\mu_{2^{-m}}\left(A_{2^{-k}}\right) - \mu_{2^{-m}}\left(A_{2^{-k-1}}\right)\right]. \tag{15}$$

By Lemma 16 we get

$$u_{2^{-m}}(0) \geq c_{18} \sum_{k=m}^{\infty} 2^{k(n-2)} \prod_{i=1}^{n} 2^{k\alpha_i/2}\left[\mu_{2^{-m}}\left(A_{2^{-k}}\right) - \mu_{2^{-m}}\left(A_{2^{-k-1}}\right)\right]. \tag{16}$$

Now apply Abel's summation formula

$$u_{2^{-m}}(0) \geq c_{19} \sum_{k=m+2}^{\infty} \frac{\mu_{2^{-m}}\left(A_{2^{-k}}\right)}{cap_{\Sigma}\left(\mathcal{E}_{2^{-k}}^{0}(1)\right)}. \tag{17}$$

By Lemma 25 and the Harnack's type inequality

$$u_{2^{-m}}(0) \geq \frac{c_{19}}{2} \sum_{k=m+2}^{\infty} \frac{cap_{\Sigma}\left(A_{2^{-k}}\right)}{cap_{\Sigma}\left(\mathcal{E}_{2^{-k}}^{0}(1)\right)}.$$

Since $u_{2^{-m}}(0) \to 0$ for $m \to \infty$, then

$$\lim_{m\to\infty} \sum_{k=m+2}^{\infty} \frac{cap_{\Sigma}\left(A_{2^{-k}}\right)}{cap_{\Sigma}\left(\mathcal{E}_{2^{-k}}^{0}(1)\right)} = 0,$$

i.e. $\sum_{k=0}^{\infty} \frac{cap_{\Sigma}\left(A_{2^{-k}}\right)}{cap_{\Sigma}\left(\mathcal{E}_{2^{-k}}^{0}(1)\right)} < \infty$ and the Theorem is proved.

**Corollary 8.** *For the regularity of the point $0 \in \partial D$ it is necessary and sufficient that*

$$\sum_{k=0}^{\infty} 2^{k(n-2+|\alpha|/2)} cap\left(A_{2^{-k}}\right) = \infty,$$

*where* $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Let $\chi(\tau) = \frac{Cap(A_\tau)}{Cap\left(\mathcal{E}_\tau^0(1)\right)}, \tau \in (0,d),\ d = diamD$.

**Corollary 9.** *For the regularity of the point $0 \in \partial D$, it is necessary and suficient that*

$$\int_0^d \frac{\chi(\tau)}{\tau} d\tau = \infty.$$

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