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ON THE SOLVABILITY AND ESTIMATE OF THE SOLUTION OF A BOUNDARY VALUE PROBLEM

Abstract

The continuation on parameter method is applied to the solvability and estimate of the solution of a boundary value problem for a second order differential equation with operator coefficients.

The application of the continuation in parameter method to the solvability and estimate of the solution of a boundary value problem for a second order differential equation is given in the paper.

Consider on $0 \le t \le T$ the boundary value problem

$$-\frac{d^2 x(t)}{dt^2} + A(t)\frac{dx(t)}{dt} + B(t)x(t) = f(t)\ ,\ \ x(0) = x(T) = 0\ , \tag{1}$$

where $A(t)$ and $B(t)$ at any $t \in [0,T]$ are $n \times n$-dimensional matrix-functions acting in the space $\Re^n$; $f(t)$ is a given, and $x(t)$ is a desired vector-functions with values from $\Re^n$.

Introduce the space $X = \{x(t) \in C^{(2)}([0,T],\Re^n) : x(0) = x(T) = 0$ with the norm

$$\|x\|_X = \|x\|_0 + \|x'\|_0 + \|x''\|_0,\ \ \text{where } \|x\|_0 = \max_{0\le t\le T}\|x(t)\|_n \text{ and } \|x(t)\|_n = \sqrt{\sum_{i=1}^{n} x_i^2(t)}.$$

We also introduce the space $Y = \{y(t) \in C([0,T],\Re^n)\}$ with the norm $\|y\|_Y = \|y\|_0$.

Evidently, X and Y are Banach spaces.

Let $E(t)$ be such a $n \times n$- dimensional matrix function that at each $y \in \Re^n$ $E(t)y \in Y$ and $M_y = \max_{0\le t\le T}\|E(t)y\|_n$, $\|E\|_{\Re^n \to Y} = \sup\{M_y : \|y\|_n \le 1\}$,

$$\|E\|_{X\to Y} = \sup\left\{\max_{0\le t\le T}\|E(t)x(t)\|_n : (t) \in X, \|x\|_X \le 1\right\}\ .$$

Assume that the following conditions are fulfilled:

1) $A(t)$ is a continuously differentiable on $[0,T]$ symmetric matrix-function;
2) $B(t)$ is a continuous on $[0,T]$ matrix-function;
3) $\forall t \in [0,T]$ and $\forall y \in \Re^n$ $\exists \alpha \in \Re$: $<\left(B(t) - \frac{1}{2}A'(t)\right)y, y> \ge \alpha\|y\|_n^2$, where $\alpha > -\frac{8}{\pi T^2}$ and $<\cdot,\cdot>$ is a scalar product in $\Re^n$;
4) $f(t) \in Y$.

Determine an operator function $P(\lambda)$ acting at each $\lambda \in [0,1]$ from X to Y, by the following way:

$$P(\lambda) = -I\frac{d^2}{dt^2} + \lambda A(t)\frac{d}{dt} + \lambda B(t). \tag{2}$$

Here I is a unit matrix of $n \times n$ - dimensionality.

Consider the equation

$$P(\lambda)x(t)=f(t), \tag{3}$$

where $x(t)\in X$ and $\lambda\in[0,1]$, or that the same

$$\begin{cases} -I\dfrac{d^2x(t)}{dt^2}+\lambda A(t)\dfrac{dx(t)}{dt}+\lambda B(t)x(t)=f(t), \\ x(0)=x(T)=0. \end{cases} \tag{4}$$

Theorem 1. *By fulfilling the conditions 1)-3) the following statements are valid:*

a) $P(\lambda)\in L(X,Y)$ $(\lambda\in[0,1])$;

b) $P(\lambda)\in C([0,1], L(X,Y))$;

c) $\exists P^{-1}(0)\in L(Y,X)$;

d) $\forall\lambda\in[0,1]\,\exists K=const>0\ \ \forall x\in X:\|P(\lambda)x\|_Y\geq K\|x\|_X$.

Proof. Statement a) follows from the definition of space X,Y and the action rule of the operator $P(\lambda)$.

Statement b) follows from the estimate:

$$\|P(\lambda)-P(\mu)\|_{X\to Y}=|\lambda-\mu|\sup\left\{\max_{0\leq t\leq T}\left\|A(t)\frac{dx(t)}{dt}+B(t)x(t)\right\|_n : x(t)\in X\ ,\ \|x\|_X\leq 1\right\}\leq$$

$$\leq|\lambda-\mu|\sup\left\{\max_{0\leq t\leq T}\left\|A(t)\frac{dx(t)}{dt}\right\|_n+\max_{0\leq t\leq T}\|B(t)x(t)\|_n : x(t)\in X, \|x\|_X\leq 1\right\}\leq$$

$$\leq|\lambda-\mu|\sup\left\{\|A\|_{\Re^n\to Y}\cdot\max_{0\leq t\leq T}\left\|\frac{dx(t)}{dt}\right\|_n+\|B\|_{\Re^n\to Y}\cdot\max_{0\leq t\leq T}\|x(t)\|_n : x(t)\in X,\ \|x\|_X\leq 1\right\}=$$

$$=|\lambda-\mu|\sup\left\{\|A\|_{\Re^n\to Y}\cdot\|x'\|_0+\|B\|_{\Re^n\to Y}\cdot\|x\|_0 : x(t)\in X,\ \|x\|_X\leq 1\right\}\leq$$

$$\leq|\lambda-\mu|\max\left\{\|A\|_{\Re^n\to Y},\|B\|_{\Re^n\to Y}\right\}\cdot\sup\left\{\|x'\|_0+\|x\|_0 : x(t)\in X,\ \|x\|_X\leq 1\right\}\leq$$

$$\leq|\lambda-\mu|\max\left\{\|A\|_{\Re^n\to Y},\|B\|_{\Re^n\to Y}\right.$$

Thus, at any $\lambda,\mu\in[0,1]$ we have

$$\|P(\lambda)-P(\mu)\|_{X\to Y}\leq|\lambda-\mu|\max\left\{\|A\|_{\Re^n\to Y},\|B\|_{\Re^n\to Y}\right.$$

Prove the statement c). For $\lambda=0$ the boundary value problem (4) is of the view:

$$-I\frac{d^2x(t)}{dt^2}=f(t)\ ,\qquad x(0)=X(T)=0\,.$$

For the arbitrary $f(t)\in Y$, the solution of the problem exists, it is unique and determined by the formula:

$$x(t)=-\int_0^t(t-\tau)f(\tau)d\tau+\frac{t}{T}\int_0^T(T-\tau)f(\tau)d\tau\,.$$

Thus, we prove statement c).

Now prove statement d). The equation (4) multiply scalarly by $x(t)$ and integrate the obtained equality and with respect to t in bounds from 0 to T. Then, considering the condition $x(0)=X(T)=0$ and the identities

$$<x''(t),x(t)>=(<x'(t),x(t)>)'-\|x'(t)\|_n^2,$$

$$<A(t)x'(t),x(t)>=\frac{1}{2}(<A(t)x(t),x(t)>)'-\frac{1}{2}<A'(t)x(t),x(t)>$$

(since $A(t)$ is a symmetric matrix),

$$< A(t)x'(t),x(t) > + < B(t)x(t),x(t) >= \frac{1}{2}(< A(t)x(t),x(t) >)' + < \left(B(t) - \frac{1}{2}A'(t)\right)x(t),x(t) >,$$

we get

$$\int_0^T \|x'(t)\|_n^2 dt + \lambda \int_0^T < \left(B(t) - \frac{1}{2}A'(t)\right)x(t),x(t) > dt = \int_0^T < f(t),x(t) > dt .$$

Hence by virtue of condition 3) we get

$$\int_0^T \|x'(t)\|_n^2 dt + \alpha\lambda \int_0^T \|x(t)\|_n^2 dt \le \int_0^T < f(t),x(t) > dt . \qquad (5)$$

Since $0 \le \lambda \le 1$, for $\alpha < 0$ we have $\lambda\alpha \ge \alpha$. Hence and from (5) for $\alpha < 0$ it follows

$$\int_0^T \|x'(t)\|_n^2 dt + \alpha \int_0^T \|x(t)\|_n^2 dt \le \int_0^T < f(t),x(t) > dt . \qquad (6)$$

Analogously, from (5) for $\alpha \ge 0$ we get

$$\int_0^T \|x'(t)\|_n^2 dt \le \int_0^T < f(t),x(t) > dt . \qquad (7)$$

We easily establish the following inequalities for $x(t) \in C^{(1)}([0,T],\Re^n)$ $(x(0) = x(T) = 0)$:

$$\int_0^T \|x'(t)\|_n^2 dt \ge \frac{1}{T}\|x(t)\|_n^2, \qquad (8)$$

$$\|x(t)\|_n \le \sqrt{t}\sqrt{\int_0^T \|x'(t)\|_n^2 dt} , \qquad (9)$$

$$\|x(t)\|_n \le \sqrt{T-t}\sqrt{\int_0^T \|x'(t)\|_n^2 dt} , \qquad (10)$$

We multiply (9) and (10), and get

$$\|x(t)\|_n^2 \le \sqrt{(T-t)t}\sqrt{\int_0^T \|x'(t)\|_n^2 dt} .$$

Hence, if we observe that $\int_0^T \sqrt{(T-t)\,t}\; dt = \frac{\pi T^2}{8}$ we find

$$\int_0^T \|x(t)\|_n^2 dt \le \frac{\pi T^2}{8}\int_0^T \|x'(t)\|_n^2 dt , \qquad (11)$$

Considering (11) in (6) for $\alpha < 0$ we have

$$\left(\frac{8}{\pi T^2} + \alpha\right)\int_0^T \|x(t)\|_n^2 \, dt \le \int_0^T < f(t),x(t) > dt . \qquad (12)$$

Besides, it follows from (11) and (7) that for $\alpha \ge 0$ it holds

$$\frac{8}{\pi T^2}\int_0^T \|x(t)\|_n^2 dt \le \int_0^T < f(t),x(t) > dt , \qquad (13)$$

If we associate (12) and (13) we get the validity of the inequality

$$\beta\int_0^T \|x(t)\|_n^2 dt \le \int_0^T <f(t), x(t)> dt, \qquad (14)$$

where

$$\beta = \min\left\{\frac{8}{\pi T^2} + \alpha, \frac{8}{\pi T^2}\right\} > 0,$$

We shall use the well-known estimate

$$\int_0^T <f(t), x(t)> dt \le \varepsilon \int_0^T \|x(t)\|_n^2 dt + \frac{1}{4\varepsilon}\int_0^T \|f(t)\|_n^2 dt,$$

where ε is a positive constant. Assuming $\varepsilon = \frac{\beta}{2}$ in the last inequality and considering the obtained in (14) we have

$$\beta \int_0^T \|x(t)\|_n^2 dt \le \frac{\beta}{2}\int_0^T \|x(t)\|_n^2 dt + \frac{1}{2\beta}\int_0^T \|f(t)\|_n^2 dt,$$

or that the same

$$\int_0^T \|x(t)\|_n^2 dt \le \frac{1}{\beta^2}\int_0^T \|f(t)\|_n^2 dt, \qquad (15)$$

Returning to (5) we find

$$\int_0^T \|x'(t)\|_n^2 dt \le |\alpha| \int_0^T \|x(t)\|_n^2 dt + \frac{1}{2}\int_0^T \|x(t)\|_n^2 dt + \frac{1}{2}\int_0^T \|f(t)\|_n^2 dt.$$

Taking into account the inequality (15) we get

$$\int_0^T \|x'(t)\|_n^2 dt \le \gamma \int_0^T \|f(t)\|_n^2 dt, \qquad (16)$$

where

$$\gamma = \frac{2|\alpha| + 1}{2\beta^2} + \frac{1}{2}.$$

It follows from inequalities (8) and (16) that

$$\|x(t)\|_n^2 \le T\gamma \int_0^T \|f(t)\|_n^2 dt.$$

Hence we easily get the validity of the inequality

$$\|x\|_0^2 \le T^2 \gamma \|f\|_0^2$$

or that the same

$$\|x\|_0 \le T\sqrt{\gamma}\|f\|_0. \qquad (17)$$

Now estimate $\|x'\|_0$. Assume that $A(t)$, $B(t)$, $f(t)$ and $x(t)$ are number functions given on the segment $[0,T]$. Then we have from (4) ([1], p. 160)

$$\left(x'(t)\exp\left(-\lambda\int_0^t A(\tau)d\tau\right)\right)' = (-\lambda B(t)x(t) - f(t))\exp\left(-\lambda\int_0^t A(\tau)d\tau\right). \qquad (18)$$

Since $x(t)$ is a number continuously differentiable function on $[0,T]$ and $x(0) = x(T) = 0$, then there exists such a point ξ from $(0,T)$ that $x'(\xi) = 0$. If we integrate the both hand of the inequality (18) with respect to t in the bounds from ξ to t we get

$$x'(t)=\exp\left(\lambda\int_0^t A(\tau)\,d\tau\right)\int_\xi^t[\lambda B(\tau)x(\tau)-f(\tau)]\exp\left(-\lambda\int_0^\tau A(\sigma)\,d\sigma\right)d\tau. \qquad (19)$$

It is obvious that if the estimate $\|x\|_0$ is known by $\|f\|_0$, then by using (19) we can establish the estimate $\|x'\|_0$ by $\|f\|_0$.

As we know, Roll's theorem for a vector-function generally speaking, is not valid; and we also know that passage from the vector problem (4) to the vector form (18) is not always possible.

Write the (4) in the form of n systems of linear differential equations of second order (each of them is a scalar equation) with boundary conditions:

$$\begin{cases} x_i''(t)-\lambda a_{ii}(t)x_i'(t)=\lambda\sum\limits_{j=1,j\neq i}^n a_{ij}(t)x_j'(t)+\lambda\sum\limits_{j=1}^n b_{ij}(t)x_j(t)-f_i(t), \\ x_i(0)=x_i(T)=0 \quad (i=1,2,\dots,n). \end{cases}$$

Hence we easily get

$$\begin{cases} \left(x_i'(t)\exp\left(-\lambda\int_0^t a_{ii}(\tau)d\tau\right)\right)'=\left[\lambda\sum\limits_{j=1,j\neq i}^n a_{ij}(t)x_j'(t)+\lambda\sum\limits_{j=1}^n b_{ij}(t)x_j(t)-f_i(t)\right]\times \\ \qquad\times\exp\left(-\lambda\int_0^t a_{ii}(\tau)\,d\tau\right), \\ x_i(0)=x_i(T)=0 \quad (i=1,2,\dots,n). \end{cases} \qquad (20)$$

Since each component of the vector-function $x(t)=(x_1(t),x_2(t),\dots,x_n(t))$ is continuous on $[0,T]$, differentiable on $(0,T)$ and $x_i(0)=x_i(T)$, then by Roll's theorem there exist such points $\xi_i\in(0,T)$ that $x_i'(\xi_i)=0$ $(i=1,2,\dots,n)$.

If we integrate the both hand sides of equations in (20) with respect to t in bounds from ξ_i to t we get the validity of the equalities

$$\begin{aligned} x_i'(t)=\exp\left(\lambda\int_0^t a_{ii}(\tau)d\tau\right)\int_{\xi_i}^t\left[\lambda\sum_{j=1,j\neq i}^n a_{ij}(\tau)x_j'(\tau)+\lambda\sum_{j=1}^n b_{ij}(\tau)x_j(\tau)-f_i(\tau)\right]\times \\ \times\exp\left(-\lambda\int_0^\tau a_{ii}(\sigma)\,d\sigma\right)d\tau, \end{aligned} \qquad (21)$$

where $i=1,2,\dots,n$.

Applying integration formula on the parts we get:

$$\begin{aligned} &\lambda\int_{\xi_i}^t\sum_{j=1,j\neq i}^n a_{ij}(\tau)x_j'(\tau)\exp\left(-\lambda\int_0^\tau a_{ii}(\sigma)\,d\sigma\right)d\tau=\lambda\sum_{j=1,j\neq i}^n\left[a_{ij}(t)x_j(t)\exp\left(-\lambda\int_0^t a_{ii}(\sigma)\,d\sigma\right)-\right. \\ &\left.-x_j(\xi_i)a_{ij}(\xi_i)\exp\left(-\lambda\int_0^{\xi_i}a_{ii}(\sigma)\,d\sigma\right)\right]-\lambda\sum_{j=1,j\neq i}^n\int_{\xi_i}^t\left[x_j(\tau)\left(a_{ij}'(\tau)-\lambda a_{ij}(\tau)\right)a_{ii}(\tau)\right]\times \\ &\times\exp\left(-\lambda\int_0^\tau a_{ii}(\sigma)\,d\sigma\right)d\tau. \end{aligned}$$

Taking the last equality in (21) into account, after some simplifications we get

$$x_i'(t)=\lambda\sum_{j=1,j\neq i}^n\left[x_j(t)a_{ij}(t)-x_j(\xi_i)a_{ij}(\xi_i)\exp\left(\lambda\int_{\xi_i}^t a_{ii}(\sigma)\,d\sigma\right)\right]-$$

$$-\lambda\sum_{j=1,j\neq i}^{n}\int_{\xi_i}^{t}\left[x_j(\tau)\left(a'_{ij}(\tau)-\lambda a_{ij}(\tau)a_{ii}(\tau)\right)\right]\exp\left(\lambda\int_{\tau}^{t}a_{ii}(\sigma)\,d\sigma\right)\,d\tau+$$

$$+\lambda\sum_{j=1}^{n}\int_{\xi_i}^{t}b_{ij}(\tau)x_j(\tau)\exp\left(\lambda\int_{\tau}^{t}a_{ii}(\sigma)\,d\sigma\right)d\tau-\int_{\xi_i}^{t}f_i(\tau)\exp\left(\lambda\int_{\tau}^{t}a_{ii}(\sigma)\,d\sigma\right)d\tau.$$

Therefore

$$|x'_i(t)|\le\sum_{j=1,j\neq i}^{n}|x_j(t)|\cdot|a_{ij}(t)|+\sum_{j=1,j\neq i}^{n}|x_j(\xi_i)|\cdot|a_{ij}(\xi_i)|\cdot e^{T\|a_{ii}\|_0}+\left|\int_{\xi_i}^{t}\sum_{j=1,j\neq i}^{n}|x_j(\tau)|\cdot|a_{ij}(\tau)|\,d\tau\right|\cdot e^{T\|a_{ii}\|_0}+$$

$$+\left|\int_{\xi_i}^{t}\sum_{j=1,j\neq i}^{n}|x_j(\tau)|\cdot|a_{ij}(\tau)|d\tau\right|\cdot e^{T\|a_{ii}\|_0}\cdot\|a_{ii}\|_0+\left|\int_{\xi_i}^{t}\sum_{j=1}^{n}|x_j(\tau)|\cdot|b_{ij}(\tau)|d\tau\right|\cdot e^{T\|a_{ii}\|_0}+\left|\int_{\xi_i}^{t}|f_i(\tau)|d\tau\right|\cdot e^{T\|a_{ii}\|_0}.$$

By using Cauchy-Bunyakovskii inequality we get

$$|x'_i(t)|\le\sqrt{\sum_{j=1}^{n}a_{ij}^2(t)}\cdot\sqrt{\sum_{j=1}^{n}x_j^2(t)}+e^{T\|a_{ii}\|_0}\cdot\sqrt{\sum_{j=1}^{n}a_{ij}^2(\xi_i)}\cdot\sqrt{\sum_{j=1}^{n}x_j^2(\xi_i)}+\left|\int_{\xi_i}^{t}\sqrt{\sum_{j=1}^{n}x_j^2(\tau)}\times\right.$$

$$\left.\times\sqrt{\sum_{j=1}^{n}a_{ij}'^2(\tau)}d\tau\right|\cdot e^{T\|a_{ii}\|_0}+\left|\int_{\xi_i}^{t}\sqrt{\sum_{j=1}^{n}x_j^2(\tau)}\cdot\sqrt{\sum_{j=1}^{n}a_{ij}^2(\tau)}\,d\tau\right|\cdot e^{T\|a_{ii}\|_0}\|a_{ii}\|_0+\left|\int_{\xi_i}^{t}\sqrt{\sum_{j=1}^{n}x_j^2(\tau)}\times\right.$$

$$\left.\times\sqrt{\sum_{j=1}^{n}b_{ij}^2(\tau)}d\tau\right|\cdot e^{T\|a_{ii}\|_0}+\left|\int_{\xi_i}^{t}|f_i(\tau)|d\tau\right|\cdot e^{T\|a_{ii}\|_0}.$$

Thus

$$|x'_i(t)|\le\sqrt{\sum_{j=1}^{n}a_{ij}^2(t)}\cdot\|x(t)\|_n+\sqrt{\sum_{j=1}^{n}a_{ij}^2(\xi_i)}\cdot\|x(\xi_i)\|_n\cdot e^{Ta_0}+\left|\int_{\xi_i}^{t}\sqrt{\sum_{j=1}^{n}a_{ij}'^2(\tau)}\cdot\|x(\tau)\|_n d\tau\right|\times$$

$$\times e^{Ta_0}+\left|\int_{\xi_i}^{t}\sqrt{\sum_{j=1}^{n}a_{ij}^2(\tau)}\|x(\tau)\|_n d\tau\right|\cdot a_0\cdot e^{Ta_0}+\left|\int_{\xi_i}^{t}\sqrt{\sum_{j=1}^{n}b_{ij}^2(\tau)}\cdot\|x(\tau)\|_n d\tau\right|\cdot e^{Ta_0}+$$

$$+\left|\int_{\xi_i}^{t}|f_i(\tau)|d\tau\right|\cdot e^{Ta_0},\qquad(22)$$

where $a_0=\max_{1\le i\le n}\|a_{ii}\|_0$.

Let

$$a=\sum_{i=1}^{n}\max_{0\le t\le T}\sqrt{\sum_{j=1}^{n}a_{ij}^2(t)},\quad a_1=\sum_{i=1}^{n}\max_{0\le t\le T}\sqrt{\sum_{j=1}^{n}a'_{ij}(t)},\quad b=\sum_{i=1}^{n}\max_{0\le t\le T}\sqrt{\sum_{j=1}^{n}b_{ij}^2(t)}.$$

Hence and from (22) taking into account the inequality $\|x(t)\|_n\le\|x\|_0$ we have

$$\|x'(t)\|_n\le\sum_{i=1}^{n}|x'_i(t)|\le a\|x\|_0+ae^{Ta_0}\|x\|_0+a_1Te^{Ta_0}\|x\|_0+a_0aTe^{Ta_0}\|x\|_0+$$

$$+bTe^{Ta_0}\|x\|_0+\sum_{i=1}^{n}\int_0^T|f_i(\tau)|\,d\tau\cdot e^{Ta_0},$$

or that the same

$$\|x'(t)\|_n\le\delta\|x\|_0+\sum_{i=1}^{n}\int_0^T|f_i(\tau)|\,d\tau\cdot e^{Ta_0},$$

where $\delta = a + (a + a_1 T + a_0 aT + bT)e^{T\alpha_0}$. Consequently,

$$\|x'\|_0 \le \delta \|x\|_0 + nTe^{T\alpha_0}\|f\|_0. \tag{23}$$

Comparing the inequalities (17) and (23) we get

$$\|x'\|_0 \le \delta_1 \|f\|_0, \tag{24}$$

where $\delta_1 = T\left(\delta\sqrt{\gamma} + ne^{T\alpha_0}\right)$.

It follows from (4) the validity of the inequality

$$\|x''\|_0 \le \|A\|_{X\to Y}\|x'\|_0 + \|B\|_{X\to Y}\|x\|_0 + \|f\|_0.$$

Taking into account the estimates (17) and (24) we find

$$\|x''\|_0 \le \delta_2 \|f\|_0, \tag{25}$$

where $\delta_2 = \delta_1 \|A\|_{X\to Y} + T\sqrt{\gamma}\|B\|_{X\to Y} + 1$. It follows from the inequalities (17), (24) and (25) the estimate

$$\|x\|_0 + \|x'\|_0 + \|x''\|_0 \le \delta_0 \|f\|_0,$$

or, that the same

$$\|x\|_X \le \delta_0 \|f\|_0,$$

where $\delta_0 = T\sqrt{\gamma} + \delta_1 + \delta_2$. Thus we proved that for all $x \in X$ and $\lambda \in [0,1]$ it is valid the inequality

$$\|P(\lambda)x\|_Y \ge \frac{1}{\delta_0}\|x\|_X. \tag{26}$$

Theorem 1 is proved.

We use the following statement:

The theorem (continuation in parameter ([1], p. 154)). Let $E(\lambda)$ be a continuous on $[0,1]$ operator-function (at each $\lambda \in [0,1]$ $E(\lambda) \in L(X,Y)$), and operator $E(0)$ be continuously invertible, and let there exist such a constant $\Delta > 0$ that for all $\lambda \in [0,1]$ and any $x \in X$ it is valid the inequality $\|E(\lambda)x\|_Y \ge \Delta\|x\|_X$. Then the operator $E(1)$ is continuously invertible, moreover $\|E^{-1}(1)\|_{Y\to X} \ge \Delta^{-1}$.

Theorem 1 shows that the continuation in parameter principle is applicable to the operator $P(\lambda)$ and instead of Δ here we have $\frac{1}{\delta_0}$ (see the inequality (26)).

Theorem 2. *By fulfilling the conditions 1)-4) there exists such a constant $K > 0$ that for any solution $x_\lambda(t)$ of the boundary value problem (4) $(\lambda \in [0,1])$ it is valid the inequality*

$$\|x_\lambda\|_0 + \|x'_\lambda\|_0 + \|x''_\lambda\|_0 \le K\|f\|_0$$

(the constant K doesn't depend on λ and f).

Theorem 3. *By fulfilling the conditions 1)-4) there exists a unique solution $x(t)$ of the problem (1) and in addition there exists such a constant $K > 0$ not depending on $x(t)$ and $f(t)$ that it is valid the inequality*

$$\|x_\lambda\|_0 + \|x'_\lambda\|_0 + \|x''_\lambda\|_0 \le K\|f\|_0.$$

Similar boundary value problems were investigated by different methods in many papers (see for instance [1-3]).

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