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TRANSFORMATION OPERATOR FOR A CLASS OF THE DIFFERENTIAL OPERATORS WITH A SINGULARITY

Abstract

In article, existence of transformation operators was proved for a class of Sturm-Liouville differential operators, which have singularity in finite interval and some properties of kernel of this transformation operator was investigated.

1. Introduction. For solving of the inverse problems for Sturm-Liouville differential operators as regular as singular the transformation operators have a special place. In [1] some types of transform operators for Sturm-Liouville regular operators were given. In the present paper the construction method of transform operator is given for one class of Sturm-Liouville operators with a singularity on the finite segment. In the case when Sturm-Liouville operator has a singularity of Bessel's type ($\frac{l(l-1)}{x^2}$, l is a positive entire number) on the finite segment the transform operator was constructed in [2], [3], and in the case $[0,\infty)$ it was given in [4]. When Sturm-Liouville operator has a singularity of Culon type ($\frac{A}{x}$, A is some real number) on the finite segment, the transformation operator was constructed in [5].

2. Construction of the integral equation. Let's consider Sturm-Liouville differential equation

$$-y''(x)+q(x)y(x)=\lambda^2 y(x) \qquad (0\le x\le \pi), \tag{1}$$

where $q(x)$ is a real function satisfying the condition

$$\int_0^\pi x|q(x)|dx<+\infty. \tag{2}$$

So as in the considered case the function $q(x)$ satisfies the condition (2) it means that the differential equation (1) has a singularity in point $x=0$ of the order $1\le\alpha<2$.

Let's denote by $S(x,\lambda)$ the solution of the differential equation (1) satisfying the conditions

$$S(0,\lambda)=0,\ S'(0,\lambda)=1. \tag{3}$$

Then function $S(x,\lambda)$ will satisfy the following integral equation:

$$S(x,\lambda)=\frac{\sin\lambda x}{\lambda}+\int_0^x\frac{\sin\lambda(x-t)}{\lambda}q(t)S(t,\lambda)dt. \tag{4}$$

Now let's prove that for solution $S(x,\lambda)$ of the differential equation (1) it is valid the representation:

$$S(x,\lambda)=\frac{\sin\lambda x}{\lambda}+\int_0^x K(x,t)\frac{\sin\lambda t}{\lambda}dt. \tag{5}$$

In order to function $S(x,\lambda)$ of form (5) to satisfy the equation (4), it must be fulfilled the equality:

$$\int_0^x K(x,t)\frac{\sin\lambda t}{\lambda}dt=\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)\frac{\sin\lambda t}{\lambda}dt+$$

$$+\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)\int_0^t K(t,\xi)\frac{\sin\lambda\xi}{\lambda}d\xi\,dt, \qquad (6)$$

and contrary if the function $K(x,t)$ satisfies this equality then function $S(x,\lambda)$ satisfies the equation (4), that is the solution of equation (1) for the initial conditions (3). Let's transform the right-hand side of (6) so, that it has the form of Fourier transform of some function. So as

$$\frac{\sin\lambda(x-t)}{\lambda}\cdot\frac{\sin\lambda\xi}{\lambda}=\frac{\cos(x-t-\xi)-\cos\lambda(x-t+\xi)}{2\lambda^2}=\frac{1}{2}\int_{(x-t)-\xi}^{(x-t)+\xi}\frac{\sin\lambda s}{\lambda}ds \qquad (7)$$

and for $\xi=t$

$$\frac{\sin\lambda(x-t)}{\lambda}\cdot\frac{\sin\lambda t}{\lambda}=\frac{\cos(x-2t)-\cos\lambda x}{2\lambda^2}=\frac{1}{2}\int_{x-2t}^{x}\frac{\sin\lambda s}{\lambda}ds,$$

then

$$\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)\frac{\sin\lambda t}{\lambda}dt=\int_0^x \frac{\sin\lambda(x-t)}{\lambda}\frac{\sin\lambda t}{\lambda}q(t)dt=$$

$$=\frac{1}{2}\int_0^x\left(\int_{x-2t}^{x}\frac{\sin\lambda s}{\lambda}ds\right)q(t)dt=\int_{-x}^{x}\left(\frac{1}{2}\int_{\frac{x-s}{2}}^{x}q(t)dt\right)\frac{\sin\lambda s}{\lambda}ds=$$

$$=\frac{1}{2}\int_{-x}^{x}\left(\int_{\frac{x-t}{2}}^{x}q(s)ds\right)\frac{\sin\lambda t}{\lambda}dt=\frac{1}{2}\int_0^x\left(\int_{\frac{x-t}{2}}^{x}q(s)ds\right)\frac{\sin\lambda t}{\lambda}dt-$$

$$-\frac{1}{2}\int_0^x\left(\int_{\frac{x+t}{2}}^{x}q(s)ds\right)\frac{\sin\lambda t}{\lambda}dt=\int_0^x\left(\frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}q(s)ds\right)\frac{\sin\lambda t}{\lambda}dt. \qquad (8)$$

Using the formula (7) again and going on with function $K(t,\xi)$ with zero for $|\xi|>|t|$ for all $t\in(-x,x)$

$$\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)\int_0^t K(t,\xi)\frac{\sin\lambda\xi}{\lambda}d\xi dt=\int_0^x q(t)\int_0^t K(t,\xi)\left\{\frac{1}{2}\int_{x-t-\xi}^{x-t+\xi}\frac{\sin\lambda s}{\lambda}ds\right\}d\xi\,dt=$$

$$=\frac{1}{2}\int_0^x q(t)\int_{x-2t}^{x}\left\{\int_{|s-(x-t)|}^{t}K(t,\xi)d\xi\right\}\frac{\sin\lambda s}{\lambda}ds=\frac{1}{2}\int_{-x}^{x}\left\{\int_{\frac{x-s}{2}}^{x}q(t)\int_{|s-(x-t)|}^{t}K(t,\xi)d\xi dt\right\}\frac{\sin\lambda s}{\lambda}ds=$$

$$=\frac{1}{2}\int_{-x}^{x}\left\{\int_{\frac{x-t}{2}}^{x}q(s)\int_{|t-(x-s)|}^{s}K(s,\xi)d\xi ds\right\}\frac{\sin\lambda t}{\lambda}dt=\frac{1}{2}\int_0^x\left\{\int_{\frac{x-t}{2}}^{x}q(s)\int_{|t-(x-s)|}^{s}K(s,\xi)d\xi ds\right\}\frac{\sin\lambda t}{\lambda}dt. \qquad (9)$$

From (8), (9) it follows that the equality (6) is equivalent to the equality

$$\int_0^x K(x,t)\frac{\sin\lambda t}{\lambda}dt=\int_0^x\left\{\frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}q(s)ds\right\}\frac{\sin\lambda t}{\lambda}dt+\int_0^x\left\{\frac{1}{2}\int_{\frac{x-t}{2}}^{x}q(s)\int_{|t-(x-s)|}^{s}K(s,\xi)d\xi ds\right\}\frac{\sin\lambda t}{\lambda}dt-$$

$$-\int_0^x\left\{\frac{1}{2}\int_{\frac{x+t}{2}}^{x}q(s)\int_{|t+(x-s)|}^{s}K(s,\xi)d\xi ds\right\}\frac{\sin\lambda t}{\lambda}dt.$$

Consequently, from the theorem on uniqueness for Fourier transform we obtain, that function $K(x,t)$ satisfies the integral equation

$$K(x,t)=\frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}q(s)ds+\frac{1}{2}\int_{\frac{x-t}{2}}^{x}q(s)\int_{|t-(x-s)|}^{s}K(s,\xi)d\xi ds-\frac{1}{2}\int_{\frac{x+t}{2}}^{x}q(s)\int_{|t+(x-s)|}^{s}K(s,\xi)d\xi ds. \qquad (10)$$

If the function $K(x,t)$ is equal to zero for $|t|>|x|$ and satisfies the equation (10), then the functions $S(x,\lambda)$ constructed by equation (5) are the solutions of equation (6) for all λ and contrary.

3. Proof of the existence of the of equation (10). Let's write the integral equation (10) in the following form:

$$K(x,t)=\frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}q(s)ds+\frac{1}{2}\iint_{D_x(t)}q(s)K(s,\xi)dsd\xi. \qquad (10')$$

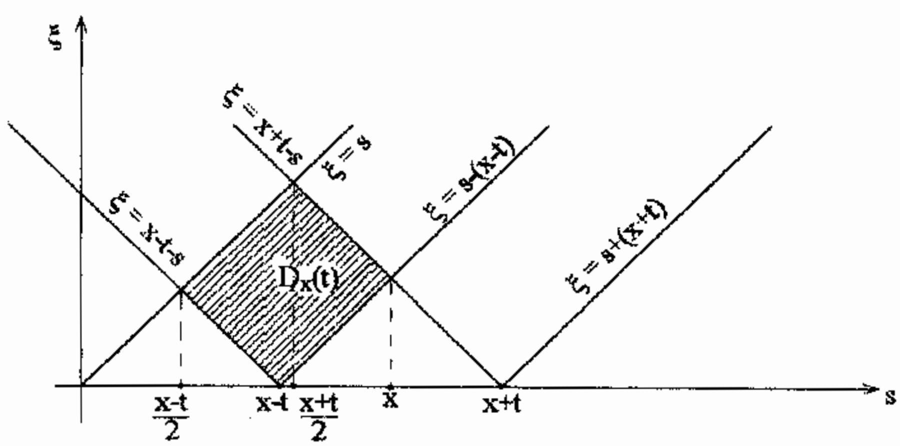

Fig. 1.

In fig. 1 the integration domain in the double integral which is in the right-hand side of equation (10'). So as for $|\xi|>|s|$ $K(s,\xi)\equiv 0$, so in fact in equation (10') the double integral should be taken only by the rectangle $D_x(t)$.

Therefore, if the solutions $S(x,\lambda)$ of equation (1) for the initial data (3) can be represented for all values of λ by equation (5), then the kernel $K(x,t)$ must satisfy the equations (10') or (10) or contrary if the function $K(x,t)$ satisfies the equation (10'), then the right-hand side of (5) for all values of λ is the solution $S(x,\lambda)$ of equation (1) for the initial data (3).

We will solve the equation (10') by the method of consequence approximations, taking

$$K_0(x,t)=\frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}} q(s)ds,$$

$$K_n(x,t)=\iint_{D_x(t)} q(s)K_{n-1}(s,\xi)dsd\xi,\quad n=1,2,\ldots.$$

So as

$$\int_0^x |K_0(x,t)|dt \le \frac{1}{2}\int_0^x dt\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}|q(s)|ds=\frac{1}{2}\int_0^{\frac{x}{2}}|q(s)|\left(\int_{x-2s}^{x}dt\right)ds+\frac{1}{2}\int_{\frac{x}{2}}^{x}|q(s)|\left(\int_{2s-x}^{x}dt\right)ds=$$

$$=\frac{1}{2}\int_0^{\frac{x}{2}}(x-x+2s)|q(s)|ds+\frac{1}{2}\int_{\frac{x}{2}}^{x}(x-2s+x)|q(s)|ds=\int_0^{\frac{x}{2}}s|q(s)|ds+\int_{\frac{x}{2}}^{x}(x-s)|q(s)|ds\le$$

$$\le\int_0^{\frac{x}{2}}s|q(s)|ds+\int_{\frac{x}{2}}^{x}s|q(s)|ds=\int_0^x s|q(s)ds|=\sigma_1(x).$$

Let's prove by the method of mathematical induction that for all $n=0,1,2,\ldots$ the following estimation is valid:

$$\int_0^x |K_n(x,t)|dt\le\frac{\{\sigma_1(x)\}^{n+1}}{(n+1)!}.$$

But if it is valid for $n-1$, then it is valid also for n as far as then

$$\int_0^x |K_n(x,t)|dt\le\frac{1}{2}\int_0^x dt\iint_{D_x(t)}|q(s)|\,|K_{n-1}(s,\xi)|dsd\xi=\frac{1}{2}\iiint_{V_x}|q(s)||K_{n-1}(s,\xi)|dsd\xi\,dt\le$$

$$\le\frac{1}{2}\int_0^x dt\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}|q(s)|ds\int_{|x-t-s|}^{s}|K_{n-1}(s,\xi)|d\xi+\frac{1}{2}\int_0^x dt\int_{\frac{x+t}{2}}^{x}|q(s)|ds\int_{|x-t-s|}^{x-s+t}|K_{n-1}(s,\xi)|d\xi\le$$

$$\le\frac{1}{2}\int_0^x dt\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}|q(s)|ds\int_0^s|K_{n-1}(s,\xi)|d\xi+\frac{1}{2}\int_0^x dt\int_{\frac{x+t}{2}}^{x}|q(s)|ds\int_0^s|K_{n-1}(s,\xi)|d\xi=$$

$$=\frac{1}{2}\int_0^{\frac{x}{2}}\left(\int_{x-2s}^{x}dt\right)|q(s)|ds\int_0^s|K_{n-1}(s,\xi)|d\xi+\frac{1}{2}\int_{\frac{x}{2}}^{x}\left(\int_{2s-x}^{x}dt\right)|q(s)|ds\int_0^s|K_{n-1}(s,\xi)|d\xi+$$

$$+\frac{1}{2}\int_{\frac{x}{2}}^{x}\left(\int_0^{2s-x}dt\right)|q(s)|ds\int_0^s|K_{n-1}(s,\xi)|d\xi\le\int_0^x s|q(s)|\left(\int_0^s|K_{n-1}(s,\xi)|d\xi\right)ds\le\frac{\{\sigma_1(x)\}^{n+1}}{(n+1)!}.$$

Thus, if $K(x,t)=\sum_{n=0}^{\infty}K_n(x,t)$, then

$$\int_0^x |K(x,t)| dt \le \sum_{n=0}^{\infty}\int_0^x |K_n(x,t)| dt \le \sum_{n=0}^{\infty}\frac{\{\sigma_1(x)\}^{n+1}}{(n+1)!} =$$
$$= \sum_{n=1}^{\infty}\frac{\{\sigma_1(x)\}^n}{n!} = e^{\sigma_1(x)} - 1 = \exp\{\sigma_1(x)\} - 1 .$$

Consequently, the integral equations (10) or (10') has the only solution $K(x,t)$.

4. Differential equation for function $K(x,t)$. If we write the integral equation (10) in the following form

$$K(x,t) = \frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}} q(s)ds + \frac{1}{2}\int_{\frac{x-t}{2}}^{x-t} qs \int_{x-t-s}^{s} K(s,\xi)d\xi\, ds + \frac{1}{2}\int_{x-t}^{x} q(s)\int_{s-x+t}^{s} K(s,\xi)d\xi\, ds -$$
$$- \frac{1}{2}\int_{\frac{x+t}{2}}^{x} q(s)\int_{x+t-s}^{t} K(s,\xi)d\xi\, ds ,$$

then differentiating twice the last equality by x and by t and subtracting each other we will obtain that function $K(x,t)$ satisfies the differential equation

$$\frac{\partial^2 K(x,t)}{\partial x^2} - \frac{\partial^2 K(x,t)}{\partial t^2} = q(x)K(x,t). \tag{11}$$

Moreover, immediately from equation (10) it follows, that

$$K(x,0) = 0 . \tag{12}$$

Now let's prove that for any $\alpha \in [1,2)$ the function $K(x,t)$ satisfies the following equality:

$$\lim_{t\to x-0}(x-t)^{\alpha-1}\left[K(x,t) - \frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}} q(s)ds\right] = 0 . \tag{13}$$

For proof of equality (13) we will need the following auxiliary confirmation.

Lemma . *Let* $f(t) \in L_1(0,a)$, $\beta > 0$. *Then*

$$\lim_{\xi\to 0}\int_{\xi}^{a} |f(t)|\left(\frac{\xi}{t}\right)^{\beta} dt = 0 .$$

Proof. Let's take $\varepsilon > 0$ and choose $\delta_0 = \delta_0(\varepsilon)$ so that

$$\int_0^{\delta_0} |f(t)| dt < \frac{\varepsilon}{2} .$$

As far as

$$\int_{\delta_0}^{a} |f(t)|\left(\frac{\xi}{t}\right)^{\beta} dt < C(\delta_0)\cdot \xi^{\beta} ,$$

then there exists such $0 < \delta < \delta_0$ that for all $\xi < \delta < \delta_0$

$$\int_{\delta_0}^{a} |f(t)|\left(\frac{\xi}{t}\right)^{\beta} dt < \frac{\varepsilon}{2} .$$

Consequently,

$$\int_{\xi}^{a}|f(t)|\left(\frac{\xi}{t}\right)^{\beta}dt \le \int_{\xi}^{\delta_0}|f(t)|dt + \int_{\delta_0}^{a}|f(t)|\cdot\left(\frac{\xi}{t}\right)^{\beta}dt < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

for all $\xi < \delta$. The lemma has been proved.

For proof of the equality (13), let's write the integral equation (10) in the following form:

$$K(x,t)=\frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}q(s)ds+\frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}q(s)\int_{x-s-t}^{s}K(s,\xi)d\xi\, ds+\frac{1}{2}\int_{x-t}^{x}q(s)\int_{s-x+t}^{s}K(s,\xi)d\xi\, ds-$$

$$-\frac{1}{2}\int_{\frac{x+t}{2}}^{x}q(s)\int_{x+t-s}^{s}K(s,\xi)d\xi\, ds.$$

Then

$$(x-t)^{\alpha-1}\left|K(x,t)-\frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}q(s)ds\right| \le \frac{1}{2}(x-t)^{\alpha-1}\int_{\frac{x-t}{2}}^{x-t}|q(s)|\int_{x-t-s}^{s}|K(s,\xi)|d\xi\, ds+$$

$$+\frac{1}{2}(x-t)^{\alpha-1}\int_{x-t}^{x}|q(s)|\int_{s-x+t}^{s}|K(s,\xi)|d\xi\, ds+\frac{1}{2}(x-t)^{\alpha-1}\int_{\frac{x+t}{2}}^{x}|q(s)|\int_{x+t-s}^{s}|K(s,\xi)|d\xi\, ds.$$

Using the properties of function $K(x,t)$ let's estimate the integrals in the right-hand side of the last inequality

$$\frac{1}{2}(x-t)^{\alpha-1}\int_{\frac{x-t}{2}}^{x-t}|q(s)|\int_{x-t-s}^{s}|K(s,\xi)|d\xi\, ds \le \frac{1}{2}(x-t)^{\alpha-1}\int_{\frac{x-t}{2}}^{x-t}|q(s)|\left|e^{\sigma_1(s)}-e^{\sigma_1(x-t-s)}\right|ds \le$$

$$\le \frac{1}{2}(x-t)^{\alpha-1}\int_{\frac{x-t}{2}}^{x-t}s|q(s)|\cdot\frac{e^{\sigma_1(s)}}{s}ds+\frac{1}{2}(x-t)^{\alpha-1}\int_{\frac{x-t}{2}}^{x-t}(x-t-s)|q(s)|\frac{e^{\sigma_1(x-t-s)}}{x-t-s}ds \le$$

$$\le \frac{1}{2}\int_{\frac{x-t}{2}}^{x-t}s|q(s)|\cdot\frac{(x-t)^{\alpha-1}}{\left(\frac{s}{e^{\sigma_1(s)}}\right)}ds+\frac{1}{2}\int_{\frac{x-t}{2}}^{x-t}s|q(s)|\cdot\frac{(x-t)^{\alpha-1}}{\left(\frac{x-t-s}{e^{\sigma_1(x-t-s)}}\right)}ds \le 2^{\alpha-1}\int_{\frac{x-t}{2}}^{x-t}s|q(s)|\cdot\left(\frac{x-t}{2^s}\right)^{\alpha-1}ds.$$

Using the lemma by virtue of condition (2) we obtain, that

$$\lim_{t\to x-0}\frac{1}{2}(x-t)^{\alpha-1}\int_{\frac{x-t}{2}}^{x-t}q(s)\int_{x-t-s}^{s}K(s,\xi)d\xi\, ds=0.$$

By analogy the second integral in the right-hand side can be estimated:

$$\frac{1}{2}(x-t)^{\alpha-1}\int_{x-t}^{x}|q(s)|\int_{s-(x-t)}^{s}|K(s,\xi)|d\xi\, ds \le \frac{1}{2}\int_{x-t}^{x}s|q(s)|\cdot\left(\frac{x-t}{s}\right)^{\alpha-1}ds.$$

By virtue of lemma we obtain, that

$$\lim_{t\to x-0}\frac{1}{2}(x-t)^{\alpha-1}\int_{x-t}^{x}|q(s)|\int_{s-(x-t)}^{s}|K(s,\xi)|d\xi\, ds=0.$$

So as the third integral for $t\to x>0$ tends to zero then we obtain

$$\lim_{t\to x-0}(x-t)^{\alpha-1}\left[K(x,t)-\frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}q(s)ds\right]=0.$$

Thus, we obtain that the function $K(x,t)$ satisfies the differential equation (11) and the conditions (12), (13).

Remark 1. If $1\le\alpha\le\frac{3}{2}$, then condition (13) can be written in the form:

$$\lim_{t\to x-0}\left[K(x,t)-\frac{1}{2}\int_{\frac{x-t}{2}}^{\frac{x+t}{2}}q(\tau)d\tau\right]=0.$$

Remark 2. If instead of (1) we consider the differential equation

$$-y''(x)+[q_0(x)+q(x)]y(x)=\lambda^2 y(x) \qquad (0\le x\le\pi), \tag{14}$$

where function $q_0(x)$ satisfies the condition (2) the real function $q(x)\in L_2[0,\pi]$, then using the above expressed method it is proved that the solution $S(x,\lambda)$ of equation (14) satisfying the conditions $S(0,\lambda)=0$, $S'(0,\lambda)=1$ can be represented in the form:

$$S(x,\lambda)=S_0(x,\lambda)+\int_0^x K(x,t)S_0(t,\lambda)dt. \tag{15}$$

Here the function $S_0(x,\lambda)$ is the solution of (14) for $q(x)=0$ satisfying the initial conditions $S_0(0,\lambda)=0$, $S_0'(0,\lambda)=1$. Moreover, the function $K(x,t)$ satisfies the integral equation of type (10), the differential equation of type (11) and the conditions of types (12), (13).

Remark 3. Particularly, if in the equation (14) put $q_0(x)=\frac{A}{x}+\frac{\delta}{x^p}$ (A, δ are real numbers, $1<p<2$), then for solution $S(x,\lambda)$ of the differential equation

$$-y''(x)+\left[\frac{A}{x}+\frac{\delta}{x^p}+q(x)\right]y(x)=\lambda^2 y(x) \qquad (0\le x\le\pi)$$

satisfying the conditions $S(0,\lambda)=0$, $S'(0,\lambda)=1$ the representation (15) is valid, where function $S_0(x,\lambda)$ is the solution of the differential equation

$$-S_0''(x,\lambda)+\left(\frac{A}{x}+\frac{\delta}{x^p}\right)S_0(x,\lambda)=\lambda^2 S_0(x,\lambda)$$

which satisfies the initial conditions: $S_0(0,\lambda)=0$, $S_0'(0,\lambda)=1$.

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