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ON THE THEORY OF NORMAL EXPANSIONS OF DIFFERENTIAL OPERATORS OF THE FIRST ORDER.

Abstract

The connection between property of formal normality of minimal operator generated by differential expression of the first order in Hilbert space of vector-functions on the finite interval, and variable operator itself, are investigated. And also all normal expressions of minimal operator in terms of boundary conditions for admissible coefficients describe.

The abstract theory of normal expansions of given non-bounded formally normal operator in Hilbert space was investigated and developed in papers of Coddington E.A. (more detailed in paper [1]). However, this theory was not applied for the theory of differential operators in Hilbert space of functions (see [2]).

We can remind, that linear close dense definite operator T in Hilbert space H is called formally normal, if $D(T)\subset D(T^*)$ and $\|Tx\|_H = \|T^*x\|_H$ for any $x\in D(T)$. Formally normal operators is called maximal, if it doesn't contains formally normal expansions. Formally normal operator, which satisfies condition $D(T)=D(T^*)$ (see [1]), is called normal operator T.

Further, by H we denote separable Hilbert space and $L_2(H,(0,1))$ is Hilbert space of vector-functions (H-valued) on finite interval. Note that all we have met integrals we understand in Lebesgue sense.

Now consider differential-operator expression of the first order of the form

$$l(u)=u'(t)+A(t)u(t),\quad 0\le t\le 1,$$

where $A(t)$ are linear close operators in H for each $t\in[0,1]$, for which:

1) $D(A(t))\cap D(A^*(t))=D$ doesn't depend on $t\in[0,1]$ and is dense in H;
2) for some real number a it is clear, that $A_R(t)\ge aE$, where $A_R(t)=\frac{1}{2}[A(t)+A^*(t)]$, E is identical operator in H;
3) vector-functions $A(t)f$ and $A^*(t)f$ for any $f\in D$ are strongly continuous on $[0,1]$ in H;
4) vector-function $A_R(t)f$ for any $f\in D$ is strongly continuously differentiable in H on $[0,1]$.

Formally adjoint differentiable expression $l(\cdot)$ in space $L_2(H,(0,1))$ have the form:

$$l^+(\mathrm{v})=-\mathrm{v}'(t)+A^*(t)\mathrm{v}(t).$$

Denote by $L_0(L_0^+)$ minimal and by $L(L^+)$ maximal operators, generated by expression $l(\cdot)$ (i.e. by adjoint expression $l^+(\cdot)$) in $L_2(H,(0,1))$ (see [3]). As a result the following conclusions are valid

$$L_0\subset L,\ L_0^+\subset L^+.$$

In present paper the connection between property of formally normality of minimal operator L_0 and operator coefficients $A(t)$ of differential expression $l(u)$ investigates, and also describes all normal expressions of minimal operator in terms of boundary conditions for admissible coefficients.

1. The following lemmas could be easily proved.

Lemma 1.1. *Let* $a(t)\in C[0,1]$. *If for each function* $\varphi(t)$ *from* $\overset{\circ}{W}{}_2^1(0,1)$ *following condition holds:*

$$\int_0^1 a(t)\varphi^2(t)dt = 0,$$

then $a(t)=0$, $0<t<1$.

By the help of this lemma, the following theorem could be proven.

Theorem 1.2. *For minimal operator* L_0 *to be formally normal in* $L_2(H,(0,1))$ *it is necessary and sufficient the validity of condition*

$$A^*(t)A(t)-A(t)A^*(t)=2A_R'(t)\,,\ \ 0<t<1. \tag{1.1}$$

From last theorem follows the

Corollary 1.3. *Let* $A(t)$ *is normal in* H *for each* $t\in[0,1]$. *Then for* L_0 *to be formally normal it is necessary and sufficiently the validity of condition*

$$A_R(t)=const\,,\ \ 0<t<1.$$

Corollary 1.4. *If* $A_R(t)=0$, $0\le t\le 1$, *then condition (1.1) holds automatically and minimal operator is antisymmetrical (i.e. formally normal).*

2. In this item we will describe all normal expansions of minimal operator L_0.

Theorem 2.1. *Each normal expansion* $\widetilde{L}$ *of minimal operator* L_0 *in space* $L_2(H,(0,1))$ *generates by differential expression* $l(u)=u'(t)+A(t)u(t)$ *and by boundary condition*

$$u(1)=Wu(0), \tag{2.1}$$

where W *is unitary operator in* H *and* $W(A_R(0)-\gamma)^{-1}=(A_R(1)-\gamma)^{-1}W$, $\gamma<a$ *is some member. Unitary operator* W *uniquely define expansion* $\widetilde{L}$, *i.e.* $\widetilde{L}=L(W)$.

Vice versa, constriction of maximal operator L *onto the set of vector-functions* $u(t)\subset D(L)$, *satisfying to condition (2.1) with some unitary operator* W *with property* $W(A_R(0)-\gamma)^{-1}=(A_R(1)-\gamma)^{-1}W$, *is a normal expansion of minimal operator in space* $L_2(H,(0,1))$.

Proof. If $\widetilde{L}$ is normal expansion of operator L_0, then operator $\widetilde{L}_J=(\widetilde{L}-\widetilde{L}^*)/(2i)$ is self- adjoint expansion of closed symmetrical minimal operator L_0^J, generated by formally symmetric differential expression $l^J(u)=-iu'+A_J(t)u$. In this case it could be proven, that triple $(\mathcal{H},\gamma_1,\gamma_2)$:

$$\mathcal{H}=H,\ \ \gamma_1(u)=[u(0)-u(1)]/\sqrt{2},\ \ \gamma_2(u)=[u(0)+u(1)]/(\sqrt{2}i)$$

is a space of boundary values of minimal operator L_0^J (see [2], [4]). Then in H there exist unique unitary operator in W such, that

$$(W-E)\gamma_1(u)+i(W+E)\gamma_2(u)=0\,,\ u(t)\in D(\widetilde{L}_J)$$

holds, here E is identical operator in H. The last statement is equivalent to the condition

$$u(1) = W u(0). \tag{2.2}$$

From the other side from second condition of formally normality it follows, that

$$(u(1), A_R(1)u(1))_H - (u(0), A_R(0)u(0))_H = 0, \; u(t) \in D(\widetilde{L}). \tag{2.3}$$

From condition (2.3), taking account of (2.2), we have

$$\left\|(A_R(1)-\gamma)^{1/2} u(1)\right\|_H = \left\|(A_R(0)-\gamma)^{1/2} u(0)\right\|_H, \quad \gamma < a.$$

From last it follows that there exists unitary operator V in H such that

$$u(1) = (A_R(1)-\gamma)^{-1/2} V (A_R(0)-\gamma)^{-1/2} u(0), \; u(t) \in D(\widetilde{L}). \tag{2.4}$$

From all said-above, it became clear that self-adjoint expansion $\widetilde{L}_J$ is determined with two boundary conditions (2.2) and (2.4). So as each self-adjoint expansion is determined only by one unitary operator W (see [4]), then $W = W_1$, i.e.

$$V = (A_R(1)-\gamma)^{1/2} W (A_R(0)-\gamma)^{1/2}.$$

Further, from relation $VV^* = E$, we found

$$W = (A_R(0)-\gamma)^{-1} = (A_R(1)-\gamma)^{-1} W.$$

Vice versa. Suppose now that W is unitary operator in H with property $W = (A_R(0)-\gamma)^{-1} = (A_R(1)-\gamma)^{-1} W$. Denote by $L(W)$ the constriction of maximal operator L onto the set of vector-functions, satisfying to condition (2.1). It is clear, that $L_0 \subset L(W) \subset L$.

Adjoint expansion $L^*(W)$ generates by formally adjoint differential expression $l^+(v) = -v'(t) + A^*(t)u(t)$ and by boundary condition $V(0) = W^* v(1)$ in space $L_2(H,(0,1))$.

Further, without any difficulties the normality of expansion $L(W)$ could be proven.

References

[1]. Coddington E.A. *Extension theory of formally normal and symmetric subspaces.* Mem. Amer.Math.Soc., 1973, 134, p. 1-80.

[2]. Maksudov F.G., Ismailov Z.I. *Normal boundary value problems for differential equations of higher order.* Turkish Journal of Mathematics, 1996, v.20, №2, p. 141-151.

[3]. Березанский Ю.М. *Разложения по собственным функциям самосопряженных операторов.* Киев: Наукова думка, 1965, 800с.

[4]. Горбачук В.И., Горбачук М.Л. *Граничные задачи для дифференциально-операторных уравнений.* Киев: Наукова думка, 1984, 284с.

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