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BOUNDARY PROPERTIES OF SOLUTIONS OF THE SECOND ORDER PARABOLIC EQUATIONS IN NON -CYLINDRICAL DOMAINS

Abstract

In present paper the first boundary value problem for parabolic equations of the second order in non-divergence form with continuous coefficients is considered. It has shown, that if coefficients satisfy the uniform Dini condition by space variables, then boundary point will be regular with respect to first boundary value problem if and only if this boundary point is regular for the heat equation.

Introduction. Denote by R_{n+1} the $(n+1)$ - dimensional Euclidean space of points $(x,t)=(x_1,\dots,x_n,t)$ and let D be bounded domain in R_{n+1}, $\Gamma(D)$ is its parabolic boundary (see [1]), $(0,0)\in\Gamma(D)$. Consider in D the first boundary value problem

$$Lu=\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_ix_j}-u_t=0,\ \ (x,t)\in D, \tag{1}$$

$$u\big|_{\Gamma(D)}=f,\ \ f\in C(\Gamma(D)), \tag{2}$$

with supposition, that $\|a_{ij}(x,t)\|$ is real symmetric matrix, moreover for $(x,t)\in D$ and any n - dimensional vector ξ the following condition holds

$$\alpha|\xi|\le\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j\le\alpha^{-1}|\xi|^2,\qquad \alpha\in(0,1]. \tag{3}$$

Further everywhere we will suppose, that generalized solution $u_f(x,t)$ of the problem (1) - (2) in sense of Wiener - Landis exists [2]. For this, for example, it is sufficient for coefficients of operator L to satisfy the uniform Dini condition in each strictly inner subdomain of domain D.

Boundary point (0,0) is called regular with respect to the first boundary value problem for equation (1), if for any continuous on $\Gamma(D)$ function f

$$\lim_{\substack{(x,t)\to(0,0)\\(x,t)\in D}}u_f(x,t)=f(0,0).$$

The regularity criterion of boundary point in terms of divergence of series of heat potentials for multidimensional heat equation for the first time was obtained by E.M.Landis [2]. In our opinion, the criterion of Landis is a full analogue of one of the form of corresponding Wiener's criterion for Laplace's equation. Nevertheless, the problem of obtaining the equivalent criterion in terms of heat capacities was of the most interest.

During the long time period the attempts to obtain such criterion were failed, although very effective from the standpoint of their verification separately necessary and sufficient conditions of regularity in capacitary terms were obtained [3-8]. Finally, L.C.Evans and R.F.Gariepy in [9] obtained above-mentioned criterion for heat equation, which can be considered as full analogue of classical form of criterion. In [10-12] the criterion of Landis was extended for the class of parabolic equations in non-divergence form with variable coefficients. The main result of these papers is the following: if coefficients of operator L satisfy the uniform Dini condition by aggregate of variables, then regularity conditions of the boundary point for equation (1) and the heat equation coincides. At present paper it has shown, that for very wide class of domains the above - mentioned fact is valid for weakening of Dini condition by time variable for condition of uniform continuity.

For the other researches concerning the problem of regularity of boundary point for parabolic equation of the second order we can mark papers [13-15].

1^0. Denotations, definitions and auxiliary facts.

For n-dimensional vector x^0, scalars $R>0$, t^1 and t^2 $\left(t^1<t^2\right)$ by $C_{x^0,R}^{t^1,t^2}$ we will denote cylinder $\left\{(x,t):\left|x-x^0\right|<R,\quad t^1<t<t^2\right\}$. Under parabolic δ-neighborhood of the point $\left(x^0,t^0\right)$, i.e. $0_\delta\left(x^0,t^0\right)$, we will understand cylinder $C_{x^0,\sqrt{\delta}}^{t^0-\delta,t^0}$. We will suppose for coefficients of operator L that following conditions hold

$$a_{ij}(x,t)\in C\left[\overline{D\cap 0_\delta(0,0)}\right];\ \left|a_{ij}(x,t)-a_{ij}(y,t)\right|\leq\varphi(|x-y|),\ \int\limits_0^{2\sqrt{\delta}}\frac{\varphi(z)}{z}dz<\infty, \tag{4}$$

where $i,j=1,\ldots,n;$ $(x,t)\in D\cap 0_\delta(0,0),$ $(y,t)\in\Gamma(D)\cap 0_\delta(0,0);$ $\varphi(z)$ is non-negative non-decreasing function on $\left(0,2\sqrt{\delta}\right)$, and

$$a_{ij}(y,\tau)=\delta_{ij}=\begin{cases}1, & if \quad i=j\\ 0, & if \quad i\neq j\end{cases}\quad for\ (y,\tau)\in\Gamma(D)\cap 0_\delta(0,0). \tag{5}$$

If there exists $\delta>0$ such, that for all $\tau\in(-\delta,0)$, $(y,\tau)\in\Gamma(D)$ the segment $\{(z,t): z=y,\ \ -\tau\leq t\leq 0\}$ doesn't intersect D, then we will say that D is R-domain in the neighborhood of the point $(0,0)$.

The simplest example of R-domain is figure of rotation $\left\{(x,t):|x|^2<\alpha(-t),-\delta<t<0\right\}$, where $\alpha(z)$ is increasing continuous on $[0,\delta]$ function.

Further everywhere, we will suppose that D is R-domain in the neighborhood of the point (0,0). Function $u(x,t)\in C^{2,1}(D)$ is called L - subparabolic (L - *superparabolic*) in D, if $Lu(x,t)\geq 0$ $(Lu(x,t)\leq 0)$ for $(x,t)\in D$.

Let

$$G(x,t)=\begin{cases}t^{-\frac{n}{2}}\ \exp\left[\frac{|x|^2}{4t}\right], & if\ \ t>0,\\ 0 & ,\ if\ \ t\leq 0.\end{cases}$$

We will use fact, which was proved in [16].

Lemma 1. Let conditions (3)-(5) hold with respect to coefficients of operator L. Then there exist functions $\Phi^{\pm}(x,y;t,\tau)$ such that if $(x,t)\in D\cap 0_\delta(0,0)$, $(y,\tau)\in\Gamma(D)\cap 0_\delta(0,0)$, $\delta<1,\ t>\tau$, then

$L_{(x,t)}\Phi^+\geq 0;$

$$\left.\begin{array}{l} C_1(t-\tau)^{1/2}, \text{ if } |x-y|^2 > 2(n+1)(t-\tau)\ln\frac{1}{t-\tau} \\ C_2G(x-y,\ t-\tau), \text{ if } |x-y|^2 \le 2(n+1)(t-\tau)\ln\frac{1}{t-\tau} \end{array}\right\} \le \Phi^+ \le C_3G(x-y,\ t-\tau);$$

and

$L_{(x,t)}\Phi^- \le 0;$

$$C_4G(x-y,t-\tau) \le \Phi^- \le \begin{cases} C_5(t-\tau)^{1/2}, & \text{if } |x-y|^2 > 2(n+1)(t-\tau)\ln\frac{1}{t-\tau} \\ C_6G(x-y,\ t-\tau), & \text{if } |x-y|^2 \le 2(n+1)(t-\tau)\ln\frac{1}{t-\tau} \end{cases}$$

For this, positive constants C_i, $i=1,\dots,6$ depend only on coefficients of operator L, n and domain D.

Everywhere further the record $C(\dots)$ means, that positive constant C depends only on contents of brackets. Introduce following sequence of positive numbers: $\tau_1 = e^{-2}$, $\tau_{m+1} = \dfrac{\tau_m}{\ln\frac{1}{\tau_m}}$ for $m \ge 1$.

Lemma 2. *The sequence $\{\tau_m\}$ is uniformly decreasing, moreover* $\lim_{m\to\infty}\tau_m = 0$.

Proof. At first, we will show that sequence $\{\tau_m\}$ is decreasing one. We have $\dfrac{\tau_2}{\tau_1} = \dfrac{1}{2} < 1$. Let for $m=k$ $\dfrac{\tau_{m+1}}{\tau_m} < 1$. We will prove that this inequality is true for $m=k+1$. From supposition $\dfrac{\tau_{k+1}}{\tau_k} < 1$ follows that $\tau_k < e^{-1}$. Therefore $\tau_{k+1} = \dfrac{\tau_k}{\ln\frac{1}{\tau_k}} < e^{-1}$. Thus, $\dfrac{\tau_{k+2}}{\tau_{k+1}} = \dfrac{1}{\ln\frac{1}{\tau_{k+1}}} < 1$, and decreasing of $\{\tau_m\}$ is proved. On the other hand

$$\tau_{m+1} = \frac{\tau_m}{\tau_{m-1}}\frac{\tau_{m-1}}{\ln\frac{1}{\tau_m}} = \frac{\tau_{m-1}}{\ln\frac{1}{\tau_{m-1}}\cdot\ln\frac{1}{\tau_m}} \le \frac{\tau_{m-1}}{\ln^2\frac{1}{\tau_{m-1}}} =$$

$$= \frac{\tau_{m-2}}{\ln\frac{1}{\tau_{m-2}}\cdot\ln^2\frac{1}{\tau_{m-1}}} \le \frac{\tau_{m-2}}{\ln^3\frac{1}{\tau_{m-2}}} \le \dots \le \frac{\tau_1}{\ln^m\frac{1}{\tau_1}} = \frac{e^{-2}}{2^m},$$

and lemma is proved.

Let for natural m

$$A_m = \left\{(x,t):\ (\tau_m)^{-n/2} \le G(-x,t) \le (\tau_{m+1})^{-n/2}\right\},\ \ H_m = A_m \setminus D,$$

$$H_m^0 = \left\{(x,t):\ -\tau_{m+2} \le t \le 0\right\} \cap H_m,\ H_m^1 = \left\{(x,t):\ t \le -\tau_{m+2}\right\} \cap H_m.$$

It is easy to see, that

$$A_m = \left\{(x,t)\colon 2n(-t)\ln\frac{\tau_{m+1}}{-t} \le |x|^2 \le 2n(-t)\ln\frac{\tau_m}{-t},\quad -\tau_m \le t \le 0\right\}.$$

Let $E \subset R_{n+1}$ is compact, $(x^0,t^0) \in R_{n+1}$. We will call measure μ with support on E admissible, if $U_\mu^E(x,t) = \int_E G(x-y,t-\tau)d\mu(y,\tau) \le 1$ for $(x,t) \notin E$. The number $U(x^0,t^0) = \sup U_\mu^E(x^0,t^0)$, where exact upper boundary we will take by all admissible measures, is called the heat potential of compact E with respect to the point (x^0,t^0). The number $p(E) = \sup \mu(E)$, where again exact upper boundary is taken by all admissible measures, is called the heat capacity of the compact E.

We denote for $m = 1,2,\ldots$ by $U_m^0(0,0)$ the potential of compact H_m^0 with respect to the point (0,0).

Lemma 3. *For any domain* D

$$\sum_{m=1}^{\infty} U_m^0(0,0) < \infty. \tag{6}$$

Proof. For simplicity we restrict ourselves by case $n \ge 3$. Let μ be admissible measure such, that

$$U_\mu^{H_m^0}(0,0) \ge \frac{1}{2}U_m^0(0,0).$$

Then

$$U_m^0(0,0) \le 2U_\mu^{H_m^0}(0,0) = 2\int_{H_m^0} G(-y,-\tau)d\mu(y,\tau) \le 2(\tau_{m+1})^{-n/2}\mu(H_m^0).$$

But from the other side $\mu(H_m^0) \le p(H_m^0)$, therefore

$$U_m^0(0,0) \le 2(\tau_{m+1})^{-n/2} p(H_m^0). \tag{7}$$

We have for sufficiently big m $H_m^0 \subset C^{-\tau_{m+2},0}_{0,\sqrt{2n\tau_{m+2}\ln\frac{\tau_m}{\tau_{m+2}}}}$. Really, so as function $z\ln\frac{\tau_m}{z}$ is increasing one for $z \in \left(0,\frac{\tau_m}{e}\right)$ and $\tau_{m+2} < \tau_{m+1} = \frac{\tau_m}{\ln\frac{1}{\tau_m}} \le \frac{\tau_m}{e}$ for sufficiently big m (by virtue of lemma 2), then above-mentioned inclusion is the corollary of definition of H_m^0. Taking into account that for sufficiently big m

$$2n\tau_{m+2}\ln\frac{\tau_m}{\tau_{m+2}} \ge 2n\tau_{m+2}\ln\frac{\tau_m}{\tau_{m+1}} = 2n\tau_{m+2}\ln\ln\frac{1}{\tau_m} \ge \tau_{m+2},$$

we obtain $H_m^0 \subset C^{-2n\tau_{m+2}\ln\frac{\tau_m}{\tau_{m+2}},0}_{0,\sqrt{2n\tau_{m+2}\ln\frac{\tau_m}{\tau_{m+2}}}}$. But according to [1] $p\left(C^{-R^2,0}_{0,R}\right) = C_7(n)R^n$.

Therefore from (7) we conclude, that

$$\sum_{m=1}^{\infty} U_m^0(0,0) \le C_8(n)\sum_{m=1}^{\infty}\left(\frac{\tau_{m+2}}{\tau_{m+1}}\ln\frac{\tau_m}{\tau_{m+2}}\right)^{n/2}. \tag{8}$$

From the other side, so as $\frac{\tau_m}{\tau_{m+1}} = \ln\frac{1}{\tau_m} < \ln\frac{1}{\tau_{m+1}} < \frac{\tau_m}{\tau_{m+2}}$, we have

$$\ln\frac{\tau_m}{\tau_{m+2}} = \ln\frac{\tau_{m+1}}{\tau_{m+2}} + \ln\frac{\tau_m}{\tau_{m+1}} < 2\ln\frac{\tau_{m+1}}{\tau_{m+2}}.$$

Using last estimation in (8), we obtain

$$\sum_{m=1}^{\infty} U_m^0(0,0) \le C_9(n)\sum_{m=1}^{\infty}\left(\frac{\tau_{m+2}}{\tau_{m+1}}\ln\frac{\tau_{m+1}}{\tau_{m+2}}\right)^{n/2} = C_9\sum_{m=2}^{\infty}\left(\frac{\tau_{m+1}}{\tau_m}\ln\frac{\tau_m}{\tau_{m+1}}\right)^{n/2}.$$

Now it is enough to take into account, that

$$\frac{\tau_{m+1}}{\tau_m} = \frac{1}{\ln\frac{1}{\tau_m}} \le \frac{1}{\ln\left(e^2 2^{m-1}\right)} \le \frac{1}{(m-1)\ln 2},$$

and then

$$\sum_{m=1}^{\infty} U_m^0(0,0) \le C_{10}(n)\sum_{m=2}^{\infty}\left(\frac{\ln m}{m}\right)^{n/2} < \infty.$$

Lemma is proved.

2^0. Theorem on increasing of positive solutions.

Let for $m = 1,2,\ldots,$ $U_m^1(0,0)$ is potential of compact H_m^1 with respect to the point (0,0), $\mathbf{C}_m$ is cylinder $C_{0,\sqrt{\tau_m}}^{-\tau_m,0}$, S_m is lateral surface $\mathbf{C}_m$, Γ_m is that part of $\Gamma(D)$, which is strictly interior with respect to $\mathbf{C}_m$.

Theorem 1. *Let in domain D the coefficients of operator L, which satisfy the conditions (3)-(5), are defined, and let $u(x,t)$ be positive solution of equation (1) in D, which is continuous in $\overline{D}$ and vanishes on Γ_m. Then there exists positive constant η which depends only on coefficients of operator L and on domain D such, that if m is sufficiently big and $M_m = \sup_{D\cap\mathbf{C}_m} u$, then*

$$M_{m-1} \ge \left(1 + \eta U_m^1(0,0)\right) M_{m+3}. \tag{9}$$

Proof. We will suppose that $U_m^1(0,0) > 0$, otherwise inequality (9) is obvious. We will fix arbitrary $\varepsilon \in \left(0, U_m^1(0,0)\right)$ and let measure μ_m on H_m^1 be such that

$$V_m(x,t) = \int_{H_m^1} G(x-y, t-\tau)\,d\mu_m(y,\tau) \le 1 \quad \text{for} \quad (x,t) \notin H_m^1, \tag{10}$$

$$V_m(0,0) > U_m^1(0,0) - \varepsilon. \tag{11}$$

Consider in $\mathbf{C}_{m-1}$ an auxiliary function $W_m(x,t) = \frac{1}{C_3}\int_{H_m^1} \Phi^+(x,y;\ t,\tau)\,d\mu_m(y,\tau)$, where function Φ^+ and constant C_3 correspond to lemma 1. Now we will estimate $\sup_{S_{m-1}} W_m$. With this aim we'll fix point $(y,\tau) \in H_m^1$ and x is such, that $|x| = \sqrt{\tau_{m-1}}$. According to lemma 1

$$\sup_{S_{m-1}} W_m \le \sup_{S_{m-1}} V_m . \qquad (12)$$

Now we will find that value of $t > \tau$, for which function $v(t) = G(x-y, t-\tau)$ reaches its maximum. Equating v_t to zero, we obtain $t - \tau = \frac{|x-y|^2}{2n}$. But for $(x,t) \in S_{m-1}$ $t - \tau \le \tau_m$, $|y| \le \sqrt{\frac{2n}{e}\tau_m}$. Therefore

$$|x-y| \ge \sqrt{\tau_{m-1}}\left(1 - \sqrt{\frac{2n}{e}\frac{\tau_m}{\tau_{m-1}}}\right) = \sqrt{\tau_{m-1}}\left(1 - \sqrt{\frac{2n}{e}\frac{1}{\ln\frac{1}{\tau_{m-1}}}}\right).$$

Thus for sufficiently big m

$$\frac{|x-y|^2}{2n} \ge \frac{\tau_{m-1}}{4n} \ge \tau_m . \qquad (13)$$

From increasing of $v(t)$ up to the fist maximum and (13) we obtain

$$\sup_{S_{m-1}} V_m \le (\tau_m)^{-n/2} \exp\left[-\frac{\tau_{m-1}}{8\tau_m}\right]\mu_m(H_m^1) = (\tau_m)^{-n/2} \exp\left[-\frac{\ln\frac{1}{\tau_{m-1}}}{8}\right]\mu_m(H_m^1) =$$

$$= b_m(\tau_m)^{-n/2}\mu_m(H_m^1) \le b_m U_m^1(0,0),$$

where $b_m = (\tau_{m-1})^{1/8}$. From the last estimation and (12) we conclude, that for sufficiently big m

$$\sup_{s_{m-1}} W_m \le b_m U_m^1(0,0). \qquad (14)$$

Consider now function $Z_m(x,t) = M_{m-1}\left[1 - W_m(x,t) + b_m U_m^1(0,0)\right]$.

According to (14) $Z_m\big|_{\Gamma(D)\cap S_{m-1}} \ge M_{m-1} \ge u\big|_{\Gamma(D)\cap S_{m-1}}$.

Moreover for points of $\Gamma(D)$ on low foundation of cylinder $\mathbf{C}_{m-1}$ $\Phi^+ = G = W_m = 0$, i.e. there $Z_m = M_{m-1}\left[1 + b_m U_m^1(0,0)\right] > u$. Finally, for $(x,t) \in \Gamma_{m-1}$ $u(x,t) = 0$, and $Z_m(x,t) \ge M_{m-1} b_m U_m^1(0,0) > 0$. Thus everywhere on $\Gamma(D) \cap \overline{\mathbf{C}}_{m-1}$ $Z_m \ge u$. According to lemma 1 and maximum principle the last inequality is valid for $(x,t) \in D \cap \mathbf{C}_{m-1}$ (if m is sufficiently big) and, in particular,

$$M_{m+3} \le M_{m-1}\left[1 - \inf_{D\cap\mathbf{C}_{m+3}} W_m + b_m U_m^1(0,0)\right]. \qquad (15)$$

Estimate now $\inf_{D\cap\mathbf{C}_{m+3}} W_m$. Let $(x,t) \in \mathbf{C}_{m+3}$, $(y,\tau) \in H_m^1$. We have for any $\sigma > 0$

$$|x-y|^2 \le (1+\sigma)|y|^2 + \left(1+\frac{1}{\sigma}\right)|x|^2 \le 2n(1+\sigma)(-\tau)\ln\frac{\tau_m}{-\tau} + \left(1+\frac{1}{\sigma}\right)\tau_{m+3} = i_1 + i_2 . \qquad (16)$$

But from the other side

$$i_1 \le 2n(1+\sigma)(t-\tau)\frac{-\tau}{-\tau-\tau_{m+3}}\ln\frac{1}{-\tau} \le 2n(1+\sigma)(t-\tau)\left(1+\frac{\tau_{m+3}}{\tau_{m+2}-\tau_{m+3}}\right)\ln\frac{1}{t-\tau} =$$

$$= 2n(1+\sigma)(1+d_m)(t-\tau)\ln\frac{1}{t-\tau},$$

where $d_m = \dfrac{\tau_{m+3}}{\tau_{m+2}-\tau_{m+3}}$. Moreover

$$i_2 \le \left(1+\frac{1}{\sigma}\right)(t-\tau)\frac{\tau_{m+3}}{-\tau-\tau_{m+3}} \le \left(1+\frac{1}{\sigma}\right)d_m(t-\tau)\ln\frac{1}{t-\tau}.$$

Taking into account these estimations in (16), we obtain

$$|x-y|^2 \le \left[2n(1+\sigma)(1+d_m)+\left(1+\frac{1}{\sigma}\right)d_m\right](t-\tau)\ln\frac{1}{t-\tau}.$$

Choose now $\sigma > 0$ such that

$$2n(1+\sigma)\left(1+\sigma^2\right)+\sigma(1+\sigma) \le 2(n+1).$$

So as $\lim\limits_{m\to\infty} d_m = 0$, then for sufficiently big m $d_m \le \sigma^2$. Thus $|x-y|^2 \le 2(n+1)(t-\tau)\ln\dfrac{1}{t-\tau}$, if m is sufficiently big. Therefore from lemma 1 and inequality (15) we conclude that

$$M_{m+3} \le M_{m-1}\left[1-\frac{C_2}{C_3}\inf_{D\cap \mathbf{C}_{m+3}} V_m + b_m U_m^1(0,0)\right]. \tag{17}$$

Now we will fix $(y,\tau)\in H_m^1$ and estimate $\inf\limits_{(x,t)\in \mathbf{C}_{m+3}} G(x-y,t-\tau)$. We have for $(x,t)\in \mathbf{C}_{m+3}$

$$G(x-y,t-\tau) \ge (-\tau)^{-n/2}\exp\left[-\frac{|x-y|^2}{4(t-\tau)}\right] = (-\tau)^{-n/2}\exp\left[-\frac{|y|^2}{4(t-\tau)}\right]\times$$

$$\times\left(\exp\left[-\frac{|y|^2}{4(t-\tau)}\right]\right)^{\frac{|x-y|^2}{|y|^2}-1} \ge (-\tau)^{-n/2}\exp\left[-\frac{|y|^2}{4(t-\tau)}\right]\left(\exp\left[-\frac{|y|^2}{4(t-\tau)}\right]\right)^{\frac{|x|^2+2|x||y|}{|y|^2}}. \tag{18}$$

Moreover

$$\left(\exp\left[-\frac{|y|^2}{4(t-\tau)}\right]\right)^{\frac{|x|^2+2|x||y|}{|y|^2}} = \exp\left[-\frac{|x|^2}{4(t-\tau)}\right]\exp\left[-\frac{|x||y|}{2(t-\tau)}\right] = j_1 j_2,$$

and further

$$j_1 \ge \exp\left[\frac{-d_m}{4}\right] \ge \frac{1}{2},$$

if m is sufficiently big. So as $\dfrac{-\tau}{t-\tau} \le 2$ (for sufficiently big m), then

$$j_2 \geq \exp\left[\frac{-|x||y|}{-\tau}\right] \geq \exp\left[-\sqrt{\frac{2n\tau_{m+3}\ln\frac{\tau_m}{-\tau}}{-\tau}}\right] \geq \exp\left[-\sqrt{\frac{2n\tau_{m+3}\ln\frac{\tau_m}{\tau_{m+2}}}{\tau_{m+2}}}\right] =$$

$$= \exp\left[-\sqrt{\frac{2n\ln\frac{\tau_m}{\tau_{m+2}}}{\ln\frac{1}{\tau_{m+2}}}}\right] \geq \exp\left[-2\sqrt{\frac{n\ln\frac{\tau_{m+1}}{\tau_{m+2}}}{\ln\frac{1}{\tau_{m+2}}}}\right] \geq \exp\left[-2\sqrt{n}\right].$$

Thus from (18) we obtain

$$G(x-y,\, t-\tau) \geq \exp[-2\sqrt{n}](-\tau)^{-\frac{n}{2}} \exp\left[-\frac{|y|^2}{4(t-\tau)}\right]. \tag{19}$$

From the other side

$$\exp\left[-\frac{|y|^2}{4(t-\tau)}\right] = \exp\left[-\frac{|y|^2}{4(-\tau)}\right]\left(\exp\left[-\frac{|y|^2}{4(-\tau)}\right]\right)^{\frac{-t}{t-\tau}} \geq \exp\left[-\frac{|y|^2}{4(-\tau)}\right]\left(\exp\left[-\frac{n}{2}\ln\frac{\tau_m}{-\tau}\right]\right)^{\frac{-t}{t-\tau}} \geq$$

$$\geq \exp\left[-\frac{|y|^2}{4(-\tau)}\right]\left(\frac{-\tau}{\tau_m}\right)^{\frac{n}{2}\frac{\tau_{m+3}}{-\tau-\tau_{m+3}}} = \exp\left[-\frac{|y|^2}{4(-\tau)}\right]J.$$

But for sufficiently big m

$$\ln\frac{1}{J} \leq \frac{n}{2}\frac{\tau_{m+3}}{\tau_{m+2}-\tau_{m+3}}\ln\frac{\tau_m}{\tau_{m+2}} \leq n\frac{\tau_{m+3}}{\tau_{m+2}}\ln\frac{\tau_m}{\tau_{m+2}} = n\frac{\ln\frac{\tau_m}{\tau_{m+2}}}{\ln\frac{1}{\tau_{m+2}}} \leq n.$$

Therefore from (19) we obtain $G(x-y,t-\tau) \geq \exp[-3n]G(-y,-\tau)$, and this inequality with (17) and (11) gives us

$$M_{m+3} \leq M_{m-1}\left[1-\frac{C_2}{C_3}\exp[-3n]V_m(0,0)+b_m U_m^1(0,0)\right] \leq$$

$$\leq M_{m-1}\left[1-U_m^1(0,0)\left(\frac{C_2}{C_3}\exp[-3n]-b_m\right)+\frac{C_2}{C_3}\exp[-3n]\varepsilon\right].$$

Taking into account an arbitrarity of $\varepsilon > 0$, we conclude that

$$M_{m+3} \leq M_{m-1}\left[1-U_m^1(0,0)\left(\frac{C_2}{C_3}\exp[-3n]-b_m\right)\right]. \tag{20}$$

Now it is enough to note, that $\lim\limits_{m\to\infty} b_m = 0$, i.e. for sufficiently big m $b_m \leq \frac{C_2}{2C_3}\exp[-3n]$.

Now from (20) follows required estimate (9) with $\eta = \frac{C_2}{2C_3}\exp[-3n]$. Theorem is proved.

3^0. Sufficient condition of regularity.

Now we denote for natural m the potential of compact H_m with respect to the point (0,0) by $U_m(0,0)$.

Theorem 2. *Let in the domain D coefficients of operator L, which satisfy the conditions (3)-(5), are defined. For regularity of point (0,0) with respect to the first boundary value problem for equation (1) is sufficient that*

$$\sum_{m=1}^{\infty} U_m(0,0) = \infty . \tag{21}$$

Proof. From (21) and lemma 3 it follows that

$$\sum_{m=1}^{\infty} U_m^1(0,0) = \infty .$$

Therefore, in its turn, it follows, that diverges at least one of the series $\sum_{m=1}^{\infty} U_{4m+l}^1(0,0)$, where $l = 0,1,2,3$. Suppose for certainty that the above-mentioned series diverges for $l = 1$.

For regularity of point (0,0) it is sufficient to show the following [1]: whatever the numbers $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are, whatever the subdomain D' of domain D, which entirely lies in half-space $t < 0$ be, and whatever solution $u(x,t) \le 1$ of equation (1) in D', continuous in $\overline{D'}$, be there exists $\sigma > 0$ such that from inequality

$$u\Big|_{\Gamma(D')\cap 0_{\varepsilon_1}(0,0)} \le 0$$

follows inequality

$$u\Big|_{D'\cap 0_{\sigma}(0,0)} < \varepsilon_2 .$$

Let m_1 be the least natural number such, that $\tau_{4m_1} < \varepsilon_1$ and for $m \ge m_1$ theorem 1 takes place. Let further the natural number $m^* \ge m_1$ be such that there exists point $(x',t') \in D' \cap \mathbf{C}_{4m^*}$, in which $u(x',t') \ge \varepsilon_2$. So as $\widetilde{H}_m^1 = \{(x,t): t \le -\tau_{m+2}\} \cap \cap (A_m \setminus D') \supset H_m^1$, then $\widetilde{U}(0,0) \ge U_m^1(0,0)$, where $\widetilde{U}_m(0,0)$ is potential of compact $\widetilde{H}_m^1$ with respect to the point (0,0). Therefore, applying successively theorem 1, we obtain

$$1 \ge M_{4m_1} \ge \prod_{i=m_1}^{m^*-1} \left(1 + \eta U_{4i+1}^1(0,0)\right) M_{4m^*} \ge \prod_{i=m_1}^{m^*-1} \left(1 + \eta U_{4i+1}^1(0,0)\right) \varepsilon_2 ,$$

or,

$$\sum_{i=m_1}^{m^*-1} \ln\left(1 + \eta U_{4i+1}^1(0,0)\right) \le \ln \frac{1}{\varepsilon_2} . \tag{22}$$

Taking into account now the inequalities $U_m^1(0,0) \le 1$ and $\ln(1+\eta t) \ge C_{11}(\eta) t$ for $t \in [0,1]$, we obtain from (20)

$$\sum_{i=m_1}^{m^*-1} U_{4i+1}^1(0,0) \le \frac{1}{C_{11}} \ln \frac{1}{\varepsilon_2} .$$

By virtue of divergence of series $\sum_{i=1}^{\infty} U_{4i+1}^1$, the last inequality could be valid only for $m^* \le \overline{m}$. Now it is enough to choose $\sigma = \tau_{4(\overline{m}+1)}$, and theorem is proved.

4^0. Necessary condition of regularity.

Theorem 3. *Let in domain D the coefficients of operator L, satisfying the conditions (3-5), are defined. For regularity of point (0,0) with respect to the first boundary value problem for equation (1) it is necessary the validity of condition (21).*

Proof. Suppose that condition (21) doesn't hold. Denote by m_2 the least natural number, for which

$$\sum_{m=m_2}^{\infty} U_m(0,0) \le \lambda \ , \tag{23}$$

where positive number λ will be chosen later. Now we choose continuous boundary function $f(x,t)$ of the first boundary value problem (1)-(2) such, that $f(0,0)=1,\ f(x,t)=0$ for $t \le -\tau_{m_2},\quad 0 \le f(x,t) \le 1$. Now we will give one definition.

Let $t^0 < t^1 < t^2 < \ldots < t^k$, Ω_i are bounded n- dimensional domains, F_i- are cylinders $\Omega_i \times \left(t^{i-1},\ t^i\right)$; $i = 1,\ldots,k$. The domain Q is called S-domain, if it represents as a set of interior points of union $\bigcup_{i=1}^{k} \overline{F}_i$. For this $N(Q) = \bigcup_{i=1}^{k}\left(\partial\Omega_i \times \left[t^{i-1},\ t^i\right]\right)$.

For arbitrary $(n+1)$- dimensional domain H by $\gamma(H)$ we denote aggregate of all points $(x,t) \in \partial H$, for each of which there exists $r = r(x,t)$ such, that $C_{x,r}^{t-r,t} \subset H,\ C_{x,r}^{t,t+r} \subset R_{n+1} \setminus H$. Let now $\{\varepsilon_m\}$ be the sequence of positive numbers monotonely tending to zero for $m \to \infty$, which will be defined later. According to [17] for each natural m there exist S- domain $Q_m \supset H_m$ with sufficiently smooth boundaries of foundations of compound its cylinders and measure ν_m with support in $\overline{Q}_m$, that if $\Upsilon_m(x,t) = \int_{\overline{Q}_m} G(x-y,\ t-\tau)d\nu_m(y,\tau)$, then

$$\Upsilon_m\Big|_{N(Q_m)\cup\gamma(Q_m)} = 1, \quad \Upsilon_m(0,0) \le U_m(0,0) + \varepsilon_m \,. \tag{24}$$

For this we could state, that

$$\overline{Q}_m \subset \left\{(y,\tau): |y|^2 \le 2n(-\tau)\ln\frac{\tau_{m-1}}{-\tau};\ -\tau_{m-1} \le t \le 0\right\}.$$

Consider in $D' = D \setminus \overline{\bigcup_{m=m_2}^{\infty} Q_m}$ function

$$h(x,t) = u_f(x,t) - \frac{1}{C_4}\sum_{m=m_2}^{\infty}\int_{\overline{Q}_m}\Phi^-(x,y;t,\tau)d\nu_m(y,\tau).$$

According to lemma 1 $h(x,t)$ is L- subparabolic in D'. Moreover, from the choice of boundary function $f(x,t)$ and from inequality (24) if follows that $h\big|_{\Gamma(D')} \le 0$. According to maximum principle $h(x,t) \le 0$ for $(x,t) \in D'$, i.e.

$$\varlimsup_{\substack{(x,t)\to(0,0)\\(x,t)\in D'}} u_f(x,t) \le \frac{1}{C_4}\sum_{m=m_2}^{\infty}\int_{\overline{Q}_m}\Phi^-(0,y;0,\tau)d\nu_m(y,\tau). \tag{25}$$

But from the other side for $(y,t)\in\overline{Q}_m$

$$|y|^2\le 2n(-\tau)\ln\frac{\tau_{m-1}}{-\tau}\le 2(n+1)(-\tau)\ln\frac{1}{-\tau},$$

therefore by virtue of lemma 1, (23) and (25) we conclude

$$\overline{\lim_{\substack{(x,t)\to(0,0)\\(x,t)\in D'}}}u_f(x,t)\le\frac{C_6}{C_4}\sum_{m=m_2}^{\infty}\Upsilon_m(0,0)\le\frac{C_6}{C_4}\sum_{m=m_2}^{\infty}U_m(0,0)+\frac{C_6}{C_4}\sum_{m=m_2}^{\infty}\varepsilon_m\le\frac{C_6}{C_4}\lambda+\frac{C_6}{C_4}\sum_{m=1}^{\infty}\varepsilon_m.$$

Now we choose $\lambda=\frac{C_4}{4C_6}$, $\varepsilon_m=\frac{C_4}{4C_6}2^{-m}$; $m=1,2,\ldots,$

Then

$$\overline{\lim_{\substack{(x,t)\to(0,0)\\(x,t)\in D'}}}u_f(x,t)\le\frac{1}{2},$$

i.e. the point (0,0) is irregular. Theorem is proved.

Corollary. If conditions of theorem hold, then for regularity of point (0,0) with respect to the first boundary value problem for equation (1) it is necessary and sufficient regularity of this point for the heat equation $\Delta u=u_t$. In other words, if for natural m

$\mathbf{H}_m=\left\{(x,t):\ 2^{\frac{mn}{2}}\le G(-x,-t)\le 2^{\frac{(m+1)n}{2}}\right\}\setminus D$, then for regularity of the point (0,0) with respect to the first boundary value problem for equation (1) it is necessary and sufficient that

$$\sum_{m=1}^{\infty}2^{\frac{mn}{2}}p(\mathbf{H}_m)=\infty.$$

This statement follows from theorems 2-3 and criterion of paper [9].

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