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# VARIATIONAL INEQUALITIES FOR A FOURTH ORDER QUASILINEAR HYPERBOLIC OPERATOR

**Abstract**

*A variational inequality for a class of fourth order quasilinear hyperbolic operators is considered in the paper. Theorems on the solvability «in the large» are proved.*

**0.** Theory of variational inequalities for second order quasilinear hyperbolic operators developed in papers [1], [2], [3], [4] and etc. Variational inequalities for higher order quasilinear operators are investigated only for operators with integral nonlinearity [5].

In this paper we consider a variational inequality for a class of fourth order quasilinear hyperbolic operators. The theorem on the solvability «in the large» is proved. These operators are characterized by the fact that a mixed problem for the corresponding linearized operator is correct by Petrovsky.

**1. Statement of the problem and main result.** Let $x \in \Omega, t \in (0,T)$ where $\Omega \subset R^n$ and have a sufficient smooth boundary $\Gamma = \partial\Omega$, $T > 0$.

Introduce denotations

$$(u,v)(t) = \int_\Omega u(t,x)v(t,x)dx, \ \|u(t)\|_{L^2(\Omega)} = (u,u)(t),$$

$$\widetilde{W}_p^k = \left\{ u : u \in W_p^k(\Omega), \ \Delta^i u\Big|_\Gamma = 0, \ i = 0,1,\dots,\left[\frac{k-1}{2}\right]\right\},$$

where $\left[\frac{k-1}{2}\right]$ is the whole part of $\frac{k-1}{2}$, $1 < p \le \infty$. By $K_\lambda$ and $K_0$ we denote the following convex and close sets in the spaces $\widetilde{W}_2^4$ and $\widetilde{W}_2^2$ respectively.

$$K_\lambda = \left\{ u : u \in \widetilde{W}_2^4, \ |\Delta u(x)| \le 1, \ \left|\Delta^2 u(x)\right| \le \lambda \ \text{almost everywhere (a.e) on } \Omega \right\}$$

$$K_0 = \left\{ u : u \in \widetilde{W}_2^2, \ |\Delta u(x)| \le 1 \ \text{a.e. on } \ \Omega \right\},$$

where $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Let $H_2 \subset H_1 \subset H_0$ be Hilbert spaces. We denote by $H_T(H_2,H_1,H_0)$ the following space:

$$H_T(H_2,H_1,H_0) = \{ u : u \in L_\infty(0,T;H_2), u' \in L_\infty(0,T;H_1), u'' \in L_\infty(0,T;H_0)\} .$$

We also determine the set:

$$H_T(K_\lambda,H_2,H_1,H_0) \equiv H_T(K_\lambda) =$$
$$= \{ u : u \in H_T(H_2,H_1,H_0), u_t'(t,\cdot) \in K_\lambda \ \text{a.e. on } (0,T)\}, \quad 0 < \lambda < \infty .$$

Let's consider a fourth order quasilinear operator on the domain $Q_T = (0,T) \times \Omega$

$$L(u) = u_{tt} + \Delta(a(u)\Delta u),$$

where $a(\cdot) \in C^2(R)$ and which satisfy the condition

$$a(u) \ge a_0 > 0, \quad u \in R . \tag{1}$$

We shall investigate an one-sided problem corresponding to the operator $L$ with the boundary and initial conditions

$$u(t,x)=\Delta u(t,x)=0, \quad (t,x)\in(0,T)\times\Gamma, \tag{2}$$

$$u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), \quad x\in\Omega \tag{3}$$

with the restriction

$$u_t(t,\cdot)\in K_0 \text{ a.e. on } (0,T). \tag{4}$$

The following main result is valid.

**Theorem 1.** *Let the hyperbolicity condition (1) is satisfied. Then for any* $f(\cdot)\in W_2^1(0,T;L_2(\Omega))$, $u_0\in\widetilde{W}_2^4(\Omega)\cap C^4(\overline{\Omega})$, $u_1\in K_0$ *there exists a unique function* $u(\cdot)\in H_T\left(K_0;\widetilde{W}_2^3,\widetilde{W}_2^2,L_2(\Omega)\right)$, *which satisfies the conditions (2)-(3) and the inequality*

$$(L(u),v-u_t')(t)\ge(f,v-u_t')(t) \text{ a.e. on } [0,T],$$

*where* $v\in K_0$.

**2. Solvability of a regularized one-sided problem.** First of all, we shall study the corresponding linearized problem. To this aim put $\tilde{a}(t,x)=a(u(t,x))$, where $u(\cdot)\in H_T\left(K_\lambda;\widetilde{W}_2^2,\widetilde{W}_2^1,L_2(\Omega)\right)$.

We determine the operator $B(t)$ on the space $L_2(\Omega)$ by the following equalities:

$$\begin{cases} D(B(t))=\widetilde{W}_2^4 \\ B(t)h=\Delta(a(t,x)\Delta h), \quad h\in\widetilde{W}_2^4. \end{cases}$$

It is easy to show that for any $u(\cdot)\in H_T\left(K_\lambda;\widetilde{W}_2^2,\widetilde{W}_2^1,L_2(\Omega)\right)$, $B(t)$ is a self-adjoint, positive defined operator, and the mapping $t\to B(t)h$ is twice strongly continuously differentiable.

Consider the following one-sided problem

$$\begin{cases} (w_{tt}+B(t)w-f,\ v-w_t')(t)\ge 0, \text{ a.e. on } [0,T], & (5)\\ w(0)=u_0,\ \ w_t'(0)=u_1\ , & (6) \end{cases}$$

where $v\in K_\lambda$.

We know that for any $u_0\in D(B)$, $u_1\in K_\lambda\cap D\left(B^{1/2}\right)$ the problem (5)-(6) has a unique solution $w$, where $w\in L_\infty\left(0,T;D\left(B^{1/2}\right)\right)$, $w_t'\in L_\infty\left(0,T;D\left(B^{1/2}\right)\right)$, $w_{tt}''\in L_\infty(0,T;L_2(\Omega))$, $w_t'(t,\cdot)\in K_\lambda$ a.e. on $(0,T)$ (see [6]).

Since $D\left(B^{1/2}\right)=\widetilde{W}_2^2$, we get $w\in H_T\left(K_\lambda;\widetilde{W}_2^2,\widetilde{W}_2^2,L_2(\Omega)\right)$.

Thus, we have the mappings

$$u\to w: H_T\left(K_\lambda;\widetilde{W}_2^2,\widetilde{W}_2^2,L_2(\Omega)\right)\to H_T\left(K_\lambda;\widetilde{W}_2^2,\widetilde{W}_2^2,L_2(\Omega)\right).$$

Using this, we can construct a sequence $\{u_n\}$ such that

$$u_n\in H_T\left(K_\lambda;\widetilde{W}_2^2,\widetilde{W}_2^2,L_2(\Omega)\right), \tag{7}$$

where for any $n$, $u_{n+1}$ is the solution of the following problem:

$$\left(\widetilde{L}(u_n)u_{n+1},v-u'_{n+1_t}\right)(t)\ge 0,\ v\in K_\lambda\ , \text{ a.e. on } [0,T], \tag{8}$$

$$u_{n+1}(0)=u_0,\quad u'_{n+1_t}(0)=u_1\ . \tag{9}$$

Here

$$\widetilde{L}(u)h=h_{tt}+\Delta(a(u)\Delta h)-f(t,x).$$

From (7) it follows that

$$u'_{n_t}(t,\cdot)\in K_\lambda \text{ a.e. on } (0,T).$$

Hence, we get the following priori estimates:

$$\left|\Delta u'_{n_t}(t,x)\le 1\right|,\ \left|\Delta^2 u'_{n_t}(t,x)\right|\le\lambda\text{, a.e. on }(t,x)\in Q_T \tag{10}$$

that give also the following a priori estimates

$$\left.\begin{array}{l}\left|\Delta u_n(t,x)\right|\le c,\ \left|\Delta^2 u_n(t,x)\right|\le c\lambda,\\ \left|\nabla u_n(t,x)\right|\le c,\ \left|\nabla\Delta u_n(t,x)\right|\le c\lambda,\\ \left|u_n(t,x)\right|\le c,\text{ a.e. on }(t,x)\in Q_T.\end{array}\right\} \tag{11}$$

Let's introduce the following denotations: $D_\delta h(t)=h(t+\delta)-h(t)$, $\widetilde{D}_\delta=\frac{1}{\delta}D_\delta$.

In inequality (8) put $v=u'_{n+1_t}(t+\delta,\cdot)$. Then we have

$$\left(\widetilde{L}(u_n)u_{n+1},D_\delta u'_{n+1}\right)(t)\ge 0.$$

Hence we get that

$$\left.\begin{array}{l}\left(\widetilde{L}(u_n)u_{n+1},\widetilde{D}_\delta u'_{n+1}\right)(t)\ge 0,\ \text{if }\delta>0\\ \left(\widetilde{L}(u_n)u_{n+1},\widetilde{D}_\delta u'_{n+1}\right)(t)\le 0,\ \text{if }\delta<0\end{array}\right\} \tag{12}$$

Passing to the limit for $\delta\to 0$ in inequalities (12) we get

$$\left(\widetilde{L}(u_n u_{n+1},u''_{n+1})\right)(t)=0, \tag{13}$$

hence we have:

$$\int_\Omega\left|u''_{n+1_{tt}}(t,x)\right|^2dx\le c\int_\Omega\left|\Delta\left[a(u_n(t,x)\Delta u_{n+1}(t,x))\right]\right|^2dx\le$$

$$\le\bar{c}\left(\left\|u_{n+1}(t,\cdot)\right\|^2_{W_2^4(\Omega)}+1\right). \tag{14}$$

Now, by using Lagrange's integral formula and Hölder inequality we have

$$\int_\Omega\left|\widetilde{D}_\delta u'_{n+1}(0,x)\right|^2dx=\int_\Omega\left|\int_0^1 u''_{n+1}(\tau\delta,x)d\tau\right|^2dx\le$$

$$\le\int_0^1\int_\Omega\left|u''_{n+1}(\tau\delta,x)\right|^2dxd\tau. \tag{15}$$

From (14) and (15) we get the following inequality.

$$\int_\Omega\left|\widetilde{D}_\delta u'_{n+1}(0,x)\right|^2dx\le c_1\int_0^1\left\|u_{n+1}(\tau\delta,\cdot)\right\|^2_{W_2^4(\Omega)}dx.$$

Hence we have

$$\lim_{\delta\to 0}\int_\Omega\left|\widetilde{D}_\delta u'_{n+1}(0,x)\right|^2dx\le C_1\left\|u_0(x)\right\|^2_{W_2^4(\Omega)}\le C_2, \tag{16}$$

where $c_2>0$ doesn't depend on $\lambda>0$.

Let's write the inequality (8) at the point $t+\delta$ and take $v=u'_{n+1}(t,\cdot)$. Then in the inequality (8) take $v=u'_{n+1}(t+\delta,\cdot)$.

The obtained inequalities we have the following

$$\left(D_\delta\widetilde{L}(u_n)u_{n+1},D_\delta u'_{n+1}\right)(t)\le 0\text{ a.e. on }(0,T).$$

Then we integrate the last inequality on $[0,t]$, $t\in(0,T]$:

$$\int_0^t\int_\Omega D_\delta u''_{n+1}(s,x)D_\delta u'_{n+1}(s,x)dxds+$$

$$+\int_0^t\int_\Omega D_\delta(a(u_n(s,x)))\Delta u_{n+1}(s,x)D_\delta\Delta u'_{n+1}(s,x)dxds\le$$

$$+\int_0^t\int_\Omega D_\delta f(s,x)D_\delta u'_{n+1}(s,x)dxds\ .$$

Hence (after integrating by parts) we have:

$$\frac{1}{2}\int_\Omega|D_\delta u'_{n+1}(t,x)|^2dx-\frac{1}{2}\int_\Omega|D_\delta u'_{n+1}(0,x)|^2dx+\int_0^t\int_\Omega D_\delta[a(u_n(s,x))\Delta u_{n+1}(s,x)]\times$$

$$\times D_\delta\Delta u'_{n+1}(s,x)dxds\le\int_0^t\int_\Omega|D_\delta f(s,x)|^2dxds+\int_0^t\int_\Omega|D_\delta u'_{n+1}(s,x)|^2dxds\ .$$

If we divide the both sides of last inequality to $\delta^2$ and pass to the limit for $\delta\to 0$, then we have the following inequality:

$$\frac{1}{2}\int_\Omega|u''_{n+1}(t,x)|^2dx-\frac{1}{2}\lim_{\delta\to 0}\int_\Omega\left|\widetilde{D}_\delta u'_{n+1}(0,x)\right|^2dx\le$$

$$\le\int_0^t\int_\Omega\left|\Delta[a(u_n(s,x))\Delta u_{n+1}(s,x)]'_s\right|^2dxds+\int_0^t\int_\Omega|f'_s(s,x)|^2dxds+$$

$$\le\int_0^t\int_\Omega|u''_{n+1}(s,x)|^2dxds\ .$$

Taking into account (10), (11) and (16) we obtain

$$\int_\Omega|u''_{n+1}(t,x)|^2dx\le c_1+c_2\int_0^t\int_\Omega|u''_{n+1}(s,x)|^2dxds,$$

from which, we have a priori estimate

$$\int_\Omega|u''_{n+1}(t,x)|^2dx\le c, \tag{17}$$

where $c>0$ is independent of $n$ and in generally, it depends on $\lambda>0$.

In view of existing a priori estimates (10), (11) and (17) we can select from the sequence $\{u_n\}$ such a subsequence $\{u_{nk}\}$, that

$$u_{n_k}\to u\qquad *\text{- weak in } L_\infty\left(0,T;\widetilde{W}^4_\infty(\Omega)\right), \tag{18}$$

$$u'_{n_{k_t}}\to u'_t\qquad *\text{- weak in } L_\infty\left(0,T;\widetilde{W}^4_\infty(\Omega)\right), \tag{19}$$

$$u''_{n_{k_{tt}}}\to u''_t\qquad *\text{- weak in } L_\infty\left(0,T;L_2(\Omega)\right). \tag{20}$$

In view of compactness of imbedding $W^1_\infty\left(0,T;\widetilde{W}^4_\infty(\Omega),\widetilde{W}^4_\infty(\Omega)\right)\subset$ $\subset C\left([0,T];\widetilde{W}^3_\infty(\Omega)\right)$ from (18) and (19) it follows that

$$u_{n_k}\to u\ \text{ in } C\left([0,T];\widetilde{W}^3_\infty(\Omega)\right). \tag{21}$$

Let's write the inequality (8) for $n=n_k$. Then, we can get over to limit in obtained inequalities by using (18)-(21).

If we denote the limit of the sequence $\{u_{n_k}\}$ by $u_\lambda(t,x)$, we shall get that $u_\lambda(t,x)$ is the solution of the following problem:

$$\left(\widetilde{L}(u_\lambda)u_\lambda,v-u'_{\lambda_t}\right)(t)\ge 0,\ \ v\in K_\lambda\ \text{ a.e. on } (0,T), \tag{22}$$

$$u_\lambda(0,x)=u_0(x),\ u'_{\lambda_t}(0,x)=u_1(x). \tag{23}$$

So we proved the following theorem.

**Theorem 2.** *Let the conditions of Theorem 1 are satisfied. Then problem (22)-(23) has a unique solution*

$$u(\cdot)\in H_T\left(K_\lambda;\widetilde{W}_2^4,\widetilde{W}_2^2,L_2(\Omega)\right).$$

**3. Proof of the main theorem.**

From (18)-(21) it follows that for any $\lambda>0$

$$u_\lambda\in L_\infty\left(0,T;W_\infty^4\cap\widetilde{W}_2^4\right)\cap C\left([0,T];W_\infty^3(\Omega)\right), \tag{24}$$

$$u'_{\lambda_t}\in L_\infty\left(0,T;W_\infty^4\cap\widetilde{W}_2^4\right), \tag{25}$$

$$u''_{\lambda_{tt}}\in L_\infty\left(0,T;L_2(\Omega)\right), \tag{26}$$

$$u_{\lambda_t}(t,\cdot)\in K_\lambda \text{ a.e. on } (0,T),\ \lambda>0, \tag{27}$$

from the last expression it follows that

$$\left|\Delta u'_{\lambda_t}(t,x)\right|\le 1 \text{ a.e. on } Q_T. \tag{28}$$

Hence, in particular we get the following a priori estimate

$$\left.\begin{array}{l}\left|u'_{\lambda_t}(t,x)\right|\le c,\ \left|\nabla u_{\lambda_t}(t,x)\right|\le c,\\ \left|u_\lambda(t,x)\right|\le c,\ \left|\nabla u_\lambda(t,x)\right|\le c,\ \left|\Delta u_\lambda(t,x)\right|\le c \text{ a.e. on } Q_T\end{array}\right\} \tag{29}$$

By virtue of (8) we also have a priori estimate

$$\int_\Omega\left|\widetilde{D}_\delta u'_{\lambda_t}(0,x)\right|^2dx\le c_2, \tag{30}$$

where $c_2>0$ is independent from $\delta$ and $\lambda$.

Using the similar way as it was done with done with equality (8) from (22) we will get the following estimate, for $u_\lambda(t,x)$

$$\int_0^t D_\delta\left(\widetilde{L}(u_\lambda)u_\lambda,D_\delta u'_{\lambda_t}\right)(t)ds\le 0,\ \ \delta>0.$$

Integrating by parts we have

$$\frac{1}{2}\int_\Omega\left|D_\delta u'_\lambda(t,x)\right|^2dx-\frac{1}{2}\int_\Omega\left|D_\delta u'_\lambda(0,x)\right|^2dx+$$

$$+\frac{1}{2}\int_\Omega a(u_\lambda(t,x))\left|D_\delta\Delta u_\lambda(t,x)\right|^2dx-\frac{1}{2}\int_\Omega a(u_\lambda(0,x))\left|D_\delta\Delta u_\lambda(0,x)\right|^2dx-$$

$$-\frac{1}{2}\int_0^t\int_\Omega\left(a(u_\lambda(s,x))\right)'_s\left|D_\delta\Delta u_\lambda(s,x)\right|^2dxds+$$

$$+\int_0^t\int_\Omega a(u_\lambda(s,x))\Delta u_\lambda(s+\delta,x)D_\delta\Delta u'_{\lambda_s}(s,x)dxds\le$$

$$\le\int_0^t\int_\Omega D_\delta f(s,x)D_\delta u'_\lambda(s,x)dxds. \tag{31}$$

We transform the last summand at the left hand side as:

$$J=\int_\Omega D_\delta a(u_\lambda(t,x))\Delta u_\lambda(t+\delta,x)D_\delta\Delta u_\lambda(t,x)dx-$$

$$-\int_\Omega D_\delta a(u_\lambda(0,x)\Delta u_\lambda(\delta,x)D_\delta\Delta u_\lambda(0,x))dx-\int_0^t\int_\Omega\left(D_\delta a(u_\lambda(s,x))\right)'_s\Delta u_\lambda(s+\delta,x)\times$$

$$\times_\delta\ \Delta u_\lambda(s,x)dxds-\int_0^t\int_\Omega D_\delta a(u_\lambda(s,x))\Delta u'_{\lambda_s}(s+\delta,x)D_\delta\Delta u_\lambda(s,x)dxds.$$

Taking this into account, we divide both sides of obtained inequality into $\delta^2$ and pass to the limit for $\delta \to 0$ in (31). Then we get the following inequality:

$$\frac{1}{2}\int_\Omega |u''_\lambda(s,x)|^2 dx - \frac{1}{2}\lim_{\delta\to 0}\int_\Omega \left|\widetilde{D}_\delta u'_\lambda(0,x)\right|^2 dx +$$

$$+\frac{1}{2}\int_\Omega a(u_\lambda(t,x))\left|\Delta u'_{\lambda_t}(t,x)\right|^2 dx - \frac{1}{2}\lim_{\delta\to 0}\int_\Omega a(u_\lambda(0,x))\left|\widetilde{D}_\delta \Delta u_\lambda(0,x)\right|^2 dx -$$

$$-\frac{1}{2}\int_0^t\int_\Omega a'(u_\lambda(s,x)\cdot u'_{\lambda_s}(s,x))\left|\Delta u'_{\lambda_s}(s,x)\right|^2 dxds + \int_\Omega a'(u_\lambda(t,x))u'_{\lambda_t}(t,x)\times$$

$$\times \Delta u_\lambda(t,x)\Delta u'_{\lambda_t}(t,x)dx - \lim_{\delta\to 0}\int_\Omega \widetilde{D}_\delta a(u_\lambda(0,x))\Delta u_\lambda(\delta,x)\widetilde{D}_\delta \Delta u_\lambda(0,x)dx -$$

$$-\int_0^t\int_\Omega [a''(u_\lambda(s,x))u'_\lambda(s,x) + a'(u_\lambda(s,x))u''_\lambda(s,x)]\Delta u(s,x)\Delta u'_s(s,x)dxds -$$

$$-\int_0^t\int_\Omega a'(u_\lambda(s,x))u'_{\lambda_s}(s,x)\left|\Delta u'_{\lambda_s}(s,x)\right|^2 dxds \le \int_0^t\int_\Omega |u''_\lambda(s,x)|^2 dxds +$$

$$+\int_0^t\int_\Omega |f'_s(s,x)|^2 dxds .$$

Taking into account a priori estimates (28)-(30) we get

$$\int_\Omega |u''_\lambda(t,x)|^2 dx \le c_1 + c_2\int_0^t\int_\Omega |u''_\lambda(s,x)|^2 dxds ,$$

where $c_1, c_2$ are independent of $\lambda > 0$. From the last inequality it follows that

$$\int_\Omega \left|u''_{\lambda_n}(t,x)\right|^2 dx \le c , \tag{32}$$

where $c > 0$ is independent from $\lambda > 0$.

Now determine operator $A$ on the space $L_2(\Omega)$ by the following equalities

$$\begin{cases} D(A) = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega), \\ Au = -\Delta u \,. \end{cases}$$

$A$ is a self-adjoint, positive defined operator, and for any $\mu > 0$ the operator $A$ has a resolvent $R_\mu = (I + \mu A)^{-1}$.

**Lemma.** *For any* $\lambda > 0$ *and* $\mu > 0$ *the following embedding* $R_\mu K_\mu \subset K_\lambda$ *is satisfied.*

**Proof.** Let $v = R_\mu h$, where $h \in K_\lambda$. Then $v$ be is a solution of the following boundary value problem:

$$\begin{cases} v - \mu\Delta v = h, \\ v|_\Gamma = 0. \end{cases} \tag{33}$$

From the maximum principle it follows that

$$|v|_{L_\infty(\Omega)} \le |h|_{L_\infty(\Omega)}. \tag{34}$$

As since $h \in W_2^4(\Omega)$, $h|_\Gamma = 0$ and $\Delta h|_\Gamma = 0$ we shall get from (33) that

$$\Delta v|_\Gamma = 0,\ \ \Delta v - \mu\Delta^2 v = \Delta h , \tag{35}$$

$$\Delta^2 v = 0,\ \ \Delta^2 v - \mu\Delta^3 v = \Delta^2 h. \tag{36}$$

From (34)-(36) we shall have that

$$\left.\begin{aligned}\Delta v\in W_2^2(\Omega),\\ \Delta^2 v\in W_2^2(\Omega)\end{aligned}\right\} \tag{37}$$

and besides

$$|\Delta v|_{L_\infty(\Omega)}\le|\Delta h|_{L_\infty(\Omega)}, \tag{38}$$

$$\left|\Delta^2 v\right|_{L_\infty(\Omega)}\le\left|\Delta^2 h\right|_{L_\infty(\Omega)}. \tag{39}$$

From (35)-(37) thus it follows that

$$v\in\widetilde{W}_2^4(\Omega), \tag{40}$$

and from (38) and (39) it follows

$$|\Delta v|_{L_\infty(\Omega)}\le 1,\quad \left|\Delta^2 v\right|_{L_\infty(\Omega)}\le\lambda. \tag{41}$$

From (35), (36), (40) and (41) it follows that $v\in K_\lambda$.

As since $u'_{\lambda_t}(t,\cdot)\in K_\lambda$, by virtue of Lemma $v(t)=R_\mu u'_{\lambda_t}(t,\cdot)\in K_\lambda$. Then from (22) we shall get

$$\left(\widetilde{L}(u_\lambda)u_\lambda, R_\mu u'_{\lambda_t}-u'_{\lambda_t}\right)(t)\ge 0. \tag{42}$$

It is clear that $R_\mu - I = R_\mu\left[I-R_\mu^{-1}\right]=-\mu A R_\mu$. Taking this into account in (42) we have the following inequality

$$\int_0^t\left(\widetilde{L}(u_\lambda)u_\lambda, AR_\mu u'_{\lambda_t}\right)(s)ds\le 0.$$

Hence, passing to the limit for $\mu\to 0$ we get

$$-\int_0^t\int_\Omega u''_\lambda(s,x)\Delta u'_\lambda(s,x)dxds-\int_0^t\int_\Omega\Delta\left(a(u_\lambda(s,x)\Delta u_\lambda(s,x))\Delta u'_\lambda(s,x)\right)dxds\le 0. \tag{43}$$

We can transform the second summand by the following way:

$$J=-\int_0^t\int_\Omega\nabla\left[a(u_\lambda(s,x))\nabla^3 u_\lambda(s,x)+a'(u_\lambda(s,x)\nabla u_\lambda(s,x)\Delta u_\lambda(s,x))\right]\times$$

$$\times\Delta u'_\lambda(s,x)dxds=\int_0^t\int_\Omega a(u_\lambda(s,x))\nabla^3 u_\lambda(s,x)\nabla^3 u'_\lambda(s,x)dxds-$$

$$-\int_0^t\int_\Omega\nabla\left[a'(u_\lambda(s,x))\nabla u_\lambda(s,x)\Delta u_\lambda(s,x)\right]\Delta u'_\lambda(s,x)dxds=$$

$$=\frac{1}{2}\int_\Omega a(u_\lambda(t,x))\left|\nabla^3 u_\lambda(t,x)\right|^2dx-\frac{1}{2}\int_\Omega a(u_\lambda(0,x))\left|\nabla^3 u_\lambda(0,x)\right|^2dx-$$

$$-\frac{1}{2}\int_0^t\int_\Omega a'(u_\lambda(s,x))u'_\lambda(s,x)\left|\nabla^3 u_\lambda(s,x)\right|^2dxds-$$

$$-\int_0^t\int_\Omega\left[a''(u_\lambda(s,x))\left|\nabla u_\lambda(s,x)\right|^2\Delta u_\lambda+a'(u_\lambda(s,x))\left|\Delta u_\lambda(s,x)\right|^2+\right.$$

$$\left.+a'(u_\lambda(s,x))\nabla u_\lambda(s,x)\nabla^3 u_\lambda(s,x)\right]\Delta u'_\lambda(s,x)dxds.$$

Putting this in (43), and also taking into account (28), (29) and (32) we get

$$\int_\Omega\left|\nabla^3 u_\lambda(t,x)\right|^2\le c_1+c_2\int_0^t\int_\Omega\left|\nabla^3 u_\lambda(s,x)\right|^2dxds,$$

where $c_1 > 0, c_2 > 0$ are independent from $\lambda > 0$. By applying the Grownwell's inequality we have a priori estimate

$$\int_\Omega \left|\nabla^3 u_\lambda(s,x)\right|^2 dx \le c. \tag{44}$$

By virtue of (28), (29) and (44) we can select a subsequence $\{u_{\lambda_n}\}$ such that

$$u_{\lambda_n} \to u \quad * - weak \ in \quad L_\infty\left(0,T;\widetilde{W}_2^3(\Omega) \cap W_\infty^2(\Omega)\right), \tag{45}$$

$$u'_{\lambda_n} \to u' \quad * - weak \ in \quad L_\infty\left(0,T;\widetilde{W}_\infty^2(\Omega) \cap \widetilde{W}_2^2(\Omega)\right), \tag{46}$$

$$u''_{\lambda_n} \to u'' \quad * - weak \ in \quad L_\infty\left(0,T;L_2(\Omega)\right). \tag{47}$$

From (28) and (46) it follows that

$$u'(t,\cdot) \in K_\infty \text{ a.e. on } (0,T). \tag{48}$$

It is clear that if $\lambda_1 > \lambda$, then $K_\lambda \subset K_{\lambda_1}$. Taking this into account from (22) for $\lambda = \lambda_n$ we get

$$\left(\widetilde{L}(u_{\lambda_n})u_{\lambda_n}, v - u'_{\lambda_n}\right)(t) \ge 0, \ v \in K_\lambda,$$

where $\lambda_n > \lambda$.

Here, passing to the limit for $\lambda_n \to \infty$ we get

$$\left(\widetilde{L}(u)u, \ v - u'_t\right)(t) \ge 0, \quad v \in K_\lambda, \ \lambda > 0. \tag{49}$$

Since $\overline{\bigcup_{\lambda>0} K_\lambda}\Big|_{\widetilde{W}_2^2} = K_0$ we get that the inequality (49) is valid for any $v \in K_0$.

The uniqueness of the solution is proved by standard method (see [6]).

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