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ON A BOUNDARY VALUE PROBLEM WITH SHIFT FOR LAVRENTYEV-BITSADZE EQUATION

Abstract

In the present paper the boundary value problem with shift for Lavrentyev-Bitsadze equation

$$u_{xx} + sign\, y \cdot u_{yy} = 0 \tag{1}$$

in considered in the case when on the both characteristics of equation (1) the condition is given which point-wise connects values of the solution and its derivatives simultaneously.

First, for equation (1) in the domain of hyperbolisity Ω_2 the one-valued solvability of problem with shift (1), (2), (3) is established under some assumptions for the boundary value data. Then in the mixed domain Ω for equation (1) the boundary value problem with shift (1), (12), (13) is considered. Under some assumptions for the boundary value data the uniqueness and existence of the solution of this problem are proved.

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The boundary value problems with shift for the model equations of the mixed type were formulated and solved by A.M. Nakhushev [11], they are continued and generalized by his followers [1], [5], [7], [8]. We can find the wide bibliography of works devoted to the equation of the mixed type and their appendices in monographs [2], [4], [10], [12], [13].

Let Ω be the finite one-connected domain of the plain of variables x, y, bounded by simple Jordan arc σ with the ends at the points $A(0,0)$ and $B(1,0)$, arranging on upper half plane $y>0$ and with characteristics $AC: x+y=0$ and $BC: x-y=1$ of equation (1).

Further we will denote the elliptic part of mixed domain Ω by Ω_1, and hyperbolic part by Ω_2. J is unit interval $0<x<1$ on line $y=0$. We will understand under the regular solution of equation (1) in domain Ω function $U(x,y)\in C(\overline{\Omega})\cap$ $\cap C^1(\Omega)\cap C^2(\Omega_1 \cup \Omega_2)$, satisfying equation (1) in $\Omega_1 \cup \Omega_2$ and such that $U_y(x,0)$ on the ends of interval J can be turn to infinity of the order less than unit.

First, let's consider the following boundary value problem with shift for equation (1) in domain Ω_2.

Problem 1. Find in domain Ω_2 regular solution $U(x,y)\in C(\overline{\Omega}_2)$ of equation (1), satisfying the boundary value conditions

$$U(x,0) = \tau(x)\ ,\ \ \forall x \in \bar{J}, \tag{2}$$

$$\alpha(x)U\left(\frac{x}{2},-\frac{x}{2}\right) + \alpha_1(x)\frac{d}{dx}U\left(\frac{x}{2},-\frac{x}{2}\right) + \beta_1(x)\frac{d}{dx}U\left(\frac{x+1}{2},\frac{x-1}{2}\right) = \delta(x),\ \ \forall x \in \bar{J}, \tag{3}$$

where $\alpha(x), \alpha_1(x), \beta_1(x), \delta(x), \tau(x)$ are the given functions, moreover $\tau(x), \alpha(x), \alpha_1(x), \beta_1(x), \delta(x) \in C(\bar{J}) \cap C^2(J)$.

The general regular solution of equation (1), satisfying boundary value condition (2), in domain Ω_2 can be represented in the form:

$$U(x,y) = \frac{1}{2}[\tau(x+y) + \tau(x-y)] + \frac{1}{2}\int_{x-y}^{x+y} \nu(t)dt, \tag{4}$$

where $\nu(x) \in C^1(J)$ is an arbitrary function.

Taking into account (4) in boundary value condition (3) we will obtain

$$\begin{gathered} \alpha(x)\tau(x) - \alpha(x)\int_0^x \nu(t)dt + [\alpha_1(x) + \beta_1(x)]\tau'(x) + [\beta_1(x) - \alpha_1(x)]\nu(x) = \\ = 2\delta(x) - \alpha(x)\tau(0). \end{gathered} \tag{5}$$

Taking denotation $\nu_1(x) = \int_0^x \nu(t)dt$ we will write (5) in the form:

$$[\beta_1(x) - \alpha_1(x)]\nu_1'(x) - \alpha(x)\nu_1(x) = 2\delta(x) - \alpha(x)[\tau(x) + \tau(0)] - [\alpha_1(x) + \beta_1(x)]\tau'(x). \tag{6}$$

By virtue of (4) the solvability of problem 1 is equal to the solvability of equation (6) relatively to $\nu_1(x)$, taking into account initial condition $\nu_1(0) = 0$.

Putting the found from (6) value $\nu_1'(x) = \nu(x)$ into (4) we will determine the solution of problem 1.

Let's consider the two cases of the solvability of equation (6), separately:

Case I. Let the conditions

$$\begin{gathered} \alpha_1(x) = \beta_1(x),\ \alpha(x) \neq 0,\ \forall x \in \bar{J}, \\ \alpha(0)\tau(0) + \alpha_1(0)\tau'(0) = \delta(0) \end{gathered} \tag{7}$$

be fulfilled.

Then from (6) we find immediately

$$\nu(x) \equiv \nu_1'(x) = -2\left(\frac{\delta(x)}{\alpha(x)}\right)' + \tau'(x) + 2\frac{d}{dx}\left[\frac{\alpha_1(x)}{\alpha(x)}\tau'(x)\right]. \tag{8}$$

Consequently, in this case problem 1 is one-valuedly solvable and its solution $U(x,y)$ is given by formulas (4) and (8).

Case II. Assume the conditions

$$\alpha_1(x) \neq \beta_1(x),\ \alpha(x) \in C(\bar{J}) \cap C^2(J),\ \forall x \in \bar{J} \tag{9}$$

are fulfilled.

Then we can write equation (6) in the form:

$$\nu_1'(x) + p(x)\nu_1(x) = q(x), \tag{10}$$

where

$$p(x) = \frac{\alpha(x)}{\alpha_1(x) - \beta_1(x)},\ q(x) = \frac{2\delta(x)}{\beta_1(x) - \alpha_1(x)} + \frac{\alpha_1(x) + \beta_1(x)}{\alpha_1(x) - \beta_1(x)}\tau'(x) + p(x)[\tau(x) + \tau(0)].$$

Solving equation (10), taking into account $\nu_1(0) = 0$ we will find

$$\begin{gathered} \nu(x) \equiv \nu_1'(x) = \frac{2\delta(x)}{\beta_1(x) - \alpha_1(x)} - \frac{2\beta_1(x)}{\alpha_1(x) - \beta_1(x)}p(x)\tau(x) + \frac{\alpha_1(x) + \beta_1(x)}{\alpha_1(x) - \beta_1(x)}\tau'(x) + \\ + 2p(x)\vartheta^{-1}(x)\int_0^x \frac{\delta(t)\vartheta(t)}{\alpha_1(t) - \beta_1(t)}dt + 2p(x)\vartheta^{-1}(x)\int_0^x \frac{\beta_1(t)p(t)}{\alpha_1(t) - \beta_1(t)}\vartheta(t)\tau(t)dt + \end{gathered}$$

$$+\frac{2\alpha_1(0)\tau(0)}{\alpha_1(0)-\beta_1(0)}p(x)\vartheta^{-1}(x)+2p(x)\vartheta^{-1}(x)\int_0^x\frac{\alpha_1(t)\cdot\beta_1'(t)-\alpha_1'(t)\beta_1(t)}{[\alpha_1(t)-\beta_1(t)]^2}\vartheta(t)\tau(t)dt\,, \qquad (11)$$

where

$$\vartheta(x)=\exp\left(\int_0^x p(t)dt\right),\ \vartheta^{-1}(x)=\exp\left(-\int_0^x p(t)dt\right).$$

Consequently, in this case, problem 1 is one-valuedly solvable, and its solution $U(x,y)$ is given by formula (4), (11).

Note that, formula (5) is a basic correlation between $\tau(x)$ and $\nu(x)$ on segment AB reduced from hyperbolic part Ω_2 of domain Ω.

We will consider the above described domain Ω as the finite one-connected domain in complex plane $z=x+iy$. Now let's consider the following boundary value problem with shift in mixed domain Ω.

Problem 2. Find regular solution $U(z)$ of equation (1) in domain Ω, satisfying the boundary value conditions

$$U(z)=\varphi(z),\quad \forall z\in\sigma, \qquad (12)$$

$$\alpha(x)U\left(\frac{x}{2},-\frac{x}{2}\right)+\alpha_1(x)\frac{d}{dx}U\left(\frac{x}{2},-\frac{x}{2}\right)+\beta_1(x)\frac{d}{dx}U\left(\frac{x+1}{2},\frac{x-1}{2}\right)=\delta(x)\,,\quad \forall x\in J\,, \quad (13)$$

where $\varphi(z)$, $\alpha(x)$, $\alpha_1(x)$, $\beta_1(x)$, $\delta(x)$ are given continuous functions, and

$$\alpha(x),\ \alpha_1(x),\ \beta_1(x),\ \delta(x)\in H(\bar{J})\cap C^2(J). \qquad (14)$$

Here and below $H(\bar{J})$ is the set of functions continuous by Hölder in closed interval $\bar{J}$.

For proof the uniqueness of the solution of problem 2 let's previously establish known A.V. Bitsadze extremum principal [2] with the conformity to problem 2.

The extremum principle for problem 2 is formulated so:

let the conditions:

1) $\alpha_1(t)-\beta_1(t)>0$, $\alpha(t)\ge 0$, $\beta_1(t)\le 0$, *or*
$\alpha_1(t)-\beta_1(t)<0$, $\alpha(t)\le 0$, $\beta_1(t)\ge 0$, $\forall t\in[0,1)$; (15)

2) $\alpha_1(t)\cdot\beta_1'(t)-\alpha_1'(t)\cdot\beta_1(t)\le 0$, $\forall t\in[0,1)$ (16)

are fulfilled.

Then the positive maximum and the negative minimum of solution $U(z)$ of problem 2, for $\delta(x)\equiv 0$ in closed domain $\overline{\Omega_1}$ is reached only on σ.

Indeed, using formula (4), from boundary value condition (13) it is easy to see that any solution $U(z)$ of problem 2, if it exists, satisfies the correlation

$$\begin{gathered}[\alpha_1(x)+\beta_1(x)]\tau'(x)+\alpha(x)\tau(x)-[\alpha_1(x)-\beta_1(x)]\nu(x)-\\ -\alpha(x)\cdot\int_0^x\nu(t)dt=2\delta(x)-\alpha(x)\tau(0),\ \forall x\in J\,,\end{gathered} \qquad (17)$$

where $\tau(x)=U(x,-0)$, $\nu(x)=U_y(x,-0)$.

Not loosing generality we can suppose that $\tau(0)=\varphi(0)=0$.

Assume, that $\max\limits_{\overline{\Omega}_1}U(z)=U(\zeta)>0$. It is obvious, that $\zeta\,\overline{\in}\,\Omega_1$. Let $\zeta\in J$, $\xi=\mathrm{Re}\zeta$. Then from (17) for $\delta(x)\equiv 0$ and under fulfillment of conditions (15), (16) we can write

$$\nu(\xi)=2p(\xi)\vartheta^{-1}(\xi)\int_0^{\xi}\frac{\beta_1(t)p(t)}{\alpha_1(t)-\beta_1(t)}\vartheta(t)[\tau(t)-\tau(\xi)]dt+2p(\xi)\vartheta^{-1}(\xi)\times$$
$$\times\int_0^{\xi}\frac{\alpha_1(t)\cdot\beta_1'(t)-\alpha_1'(t)\cdot\beta_1(t)}{[\alpha_1(t)-\beta_1(t)]^2}\vartheta(t)[\tau(t)-\tau(\xi)]dt-\frac{2\beta_1(0)}{\alpha_1(0)-\beta_1(0)}p(\xi)\tau(\xi)\vartheta^{-1}(\xi), \tag{18}$$

where

$$\vartheta(\xi)=\exp\left(\int_0^{\xi}p(t)dt\right),\ \vartheta^{-1}(\xi)=\exp\left(-\int_0^{\xi}p(t)dt\right),\ p(\xi)=\frac{\alpha(\xi)}{\alpha_1(\xi)-\beta_1(\xi)}.$$

Under fulfillment conditions (15) and (16) from (18) it follows that at the point $\xi,\ 0<\xi<1$ inequality $\nu(\xi)\geq 0$ has place. That contradicts to Zarembo-Jiro [3] principle, consequently, $\zeta\in\sigma$.

And if the conditions

$$\alpha_1(x)=\beta_1(x),\ \alpha(x)\neq 0,\ \alpha(x)\cdot\alpha_1(x)\leq 0,\ \forall x\in\bar{J} \tag{19}$$

are fulfilled, then from formula (17) we conclude that the above given extremum principle is an analogue of known A.V. Bitsadze extremum principle [2].

Under fulfillment of the conditions

$$\alpha_1(x)=\beta_1(x)\neq 0,\ \alpha(x)=0,\ \forall x\in\bar{J} \tag{20}$$

from (17) we can conclude that this is the principle established earlier by A.M. Nakhushev [11].

If one of three conditions: (15), (16) or (19), or (20) is fulfilled, then from the above given extremum principle the uniqueness of the solution of problem 2 follows immediately.

Let's go over to the proof of the existence of the solution of problem 2. Note that, for fulfillment of condition (20), by virtue of (17) it is equivalent to Dirichlet problem [11]

$$U(z)=\begin{cases}\varphi(z),\ \forall z\in\sigma,\\ \varphi(0)+2\int_0^{x}\delta(t)/[\alpha_1(t)+\beta_1(t)]dt,\ \forall z\in\bar{J}\end{cases}$$

for $U(z)$, which is harmonic in domain Ω_1. But if conditions (19) are fulfilled, then the existence of the solution of problem 2, in this case is divided to the two subcases

$$\alpha_1(x)=\beta_1(x)=0,\ \alpha(x)\neq 0,\ \forall x\in\bar{J}, \tag{21}$$

$$\alpha_1(x)=\beta_1(x)\neq 0,\ \alpha(x)\neq 0,\ \alpha(x)\cdot\alpha_1(x)<0,\quad \forall x\in\bar{J}. \tag{22}$$

For fulfillment of conditions (21), problem 2 is equal to Tricomy problem [2] for equation (1). But subcase (22) is a particular case of the considered below general case.

Below we will assume, that

1) curve σ coincides with normal contour σ_0:

$$\left|z-\frac{1}{2}\right|=\frac{1}{2};$$

2) function $\varphi(z)$ is represented in the form

$$\varphi(z)=x(1-x)\varphi_0(x),\ \varphi_0(x)\in C(\bar{J}); \tag{23}$$

3) functions $\alpha_1(x)$ and $\beta_1(x)$ satisfy also the condition

$$\alpha_1^2(x)+\beta_1^2(x)\neq 0,\quad \forall x\in\bar{J}. \tag{24}$$

The main correlation between $\tau(x)$ and $\nu(x)$ reduced to the segment AB from the elliptic part of domain Ω, as in case of Tricomy problem has the form [2],

$$\tau'(x)+\frac{1}{\pi}\int_0^1\left(\frac{1}{t-x}+\frac{1-2t}{t+x-2tx}\right)\nu(t)dt=f'(x), \tag{25}$$

where

$$f(x)=\frac{x(1-x)}{\pi}\int_0^1\varphi_0(t)\,t^{1/2}(1-t)^{1/2}\left[x^2-(2x-1)\,t\right]^{-1}dt.$$

By virtue of (23) the function $f(x)\in C^1(\bar{J})\cap C^\infty(J)$ also for $x\to 0,1$ turn to zero of the order not less than unit.

From formula (17) it is easy to get the following correlation

$$\begin{gathered}\tau'(x)=p_2(x)\nu(x)+2p_1(x)V_1^{-1}(x)\int_0^x\frac{\beta_1(t)V_1(t)}{\alpha_1(t)+\beta_1(t)}\nu(t)dt-2\tau(0)p_1(x)V_1^{-1}(x)+\\+\frac{2\delta(x)}{\alpha_1(x)+\beta_1(x)}-2p_1(x)V_1^{-1}(x)\int_0^x\frac{\delta(t)}{\alpha_1(t)+\beta_1(t)}V_1(t)dt,\end{gathered} \tag{26}$$

where

$$p_1(x)=\frac{\alpha(x)}{\alpha_1(x)+\beta_1(x)},\ p_2(x)=\frac{\alpha_1(x)-\beta_1(x)}{\alpha_1(x)+\beta_1(x)},$$

$$V_1(x)=\exp\left(\int_0^x p_1(t)dt\right),\ V_1^{-1}(x)=\exp\left(-\int_0^x p_1(t)dt\right).$$

Taking into account (25) and (26) it is not difficult to see now, that problem (2) is equal (in the class of the sought solutions and in mean of solvability) to the following singular integral equation:

$$a(x)\nu(x)+\frac{b(x)}{\pi i}\int_0^1\left(\frac{1}{t-x}+\frac{1-2t}{t+x-2tx}\right)\nu(t)dt=F(x), \tag{27}$$

where

$$a(x)=\alpha_1(x)-\beta_1(x),\ b(x)=i[\alpha_1(x)+\beta_1(x)], \tag{28}$$

$$F(x)=f_0(x)+f_1(x),\ f_0(x)=[\alpha_1(x)+\beta_1(x)]f'(x)-2\delta(x)+$$

$$+2\varphi(0)\alpha(x)\cdot V_1^{-1}(x)+2\alpha(x)V_1^{-1}(x)\int_0^x\frac{\delta(t)V_1(t)}{\alpha_1(t)+\beta_1(t)}dt,$$

$$f_1(x)=-\int_0^1 K(x,t)\,\nu(t)dt,$$

$$K(x,t)=\begin{cases}\dfrac{2\alpha(x)\beta_1(t)V_1(t)}{[\alpha_1(t)+\beta_1(t)]\cdot V_1(x)}, & 0\le t\le x,\\ 0, & x\le t\le 1.\end{cases}$$

On the base of (14) we conclude, that $f_0(x)$, $a(x), b(x)\in H(\bar{J})\cap C^2(J)$, and kernel $K(x,t)$ is continuously differentiable and bounded in square $0\le x,\ t\le 1$.

Let's pass to the investigation of the solvability of singular integral equation (27). Making substitution of variables ([6]):

$$\xi=\frac{x^2}{1-2x+2x^2},\quad \eta=\frac{t^2}{1-2t+2t^2}$$

and taking into account the identity:

$$\frac{1}{t-x}+\frac{1-2t}{x+t-2xt}=\frac{2t(1-t)}{t^2\left(1-2x+2x^2\right)-x^2\left(1-2t+2t^2\right)}$$

we rewrite equation (27) in the form:

$$a_1(\xi)\rho(\xi)+\frac{b_1(\xi)}{\pi i}\int_0^1\frac{\rho(\eta)d\eta}{\eta-\xi}+\int_0^1 K_1(\xi,\eta)\rho(\eta)d\eta=F_1(\xi), \tag{29}$$

where

$$a_1(\xi)=a(x),\; b_1(\xi)=b(x),\; \rho(\xi)=\left(1-2x+2x^2\right)\nu(x), \tag{29$_1$}$$

$$F_1(\xi)=\left(1-2x+2x^2\right)F(x)\;, K_1(\xi,\eta)=\frac{\left(1-2x+2x^2\right)K(x,t)}{2\sqrt{\eta(1-\eta)}},\; x=\frac{\sqrt{\xi}}{\sqrt{\xi}+\sqrt{1-\xi}}\;,$$

$$t=\frac{\sqrt{\eta}}{\sqrt{\eta}+\sqrt{1-\eta}}\,.$$

Since by virtue of (24) and (28)

$$a_1^2(\xi)-b_1^2(\xi)=a^2(x)-b^2(x)=2\left[\alpha_1^2(x)+\beta_1^2(x)\right]\neq 0\,,\quad \forall x\in\bar{J}\,,$$

then equation (29) is the equation of normal type.

In the class of functions $\nu(x)\in H(J)$, which can be turn to infinity of the order less than unit on the ends of interval J its index is equal to zero and it is solved in the proper form [9].

Equation (29) is equal (in mean of search of the solution of the pointed class) to Fredholm equation

$$\rho(\xi)+\int_0^1 K_2(\xi,\eta)\rho(\eta)d\eta=F_2(\xi), \tag{30}$$

where

$$K_2(\xi,\eta)=a_1^*(\xi)K_1(\xi,\eta)-\frac{1}{\pi i}b_1^*(\xi)Z(\xi)\int_0^1\frac{K_1(t,\eta)}{Z(t)(t-\xi)}\,,$$

$$a_1^*(\xi)=\frac{a_1(\xi)}{a_1^2(\xi)-b_1^2(\xi)}\,,\quad b_1^*(\xi)=\frac{b_1(\xi)}{a_1^2(\xi)-b_1^2(\xi)},$$

$$F_2(\xi)=a_1^*(\xi)F_1(\xi)-\frac{1}{\pi i}b_1^*(\xi)Z(\xi)\int_0^1\frac{F_1(\eta)d\eta}{Z(\eta)(\eta-\xi)},$$

$Z(\xi)=\left[a_1(\xi)+b_1(\xi)\right]X^+(\xi)=\left[a_1(\xi)-b_1(\xi)\right]X^-(\xi)$ is the canonical function.

The solvability of equation (30) follows from the uniqueness of the solution of problem 2.

By virtue of equality, solution $\rho(\xi)$ of equation (30) *is also solution of equation* (29). It is easy, to show that solution $\nu(x)$ of equation (27) expressed by formula (29_1), belongs to class $C^1(J)$.

Therefore we have proved the existence of the solution of problem 2. The abovereduced results are remained valid also in the case, when curve σ satisfies Lyapunov's condition and is finished by arcs of any small length of normal curve σ_0.

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