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BOUNDARY VALUE PROBLEMS FOR A SECOND ORDER DIFFERENTIAL EQUATION WITH A DOUBLE CHARACTERISTIC ROOT

Abstract

The equations with boundary conditions depending polynomially on the parameter are studied. The theorem on double expansion is obtained.

Consider a boundary value problem on the segment [ ]

$$y'' - (2i\lambda + p(x))y' - (\lambda^2 + q(x)\lambda + r(x))y = 0, \tag{1}$$

$$U_k(y) = \sum_{\nu=1}^{2} \alpha_{\nu k}(\lambda) y^{(\nu-1)}(0) + \beta_{\nu k}(\lambda) y^{(\nu-1)}(1) = 0, \quad k = \overline{1,2}, \tag{2}$$

where $p(x), q(x), r(x)$ are smooth functions, $\alpha_{\nu k}(\lambda), \beta_{\nu k}(\lambda)$ are such from λ polynomials that rang, $(\alpha_{\nu k}(\lambda), \beta_{\nu k}(\lambda)) = 2$ for all λ.

As we see from the equation (1), a characteristic equation has one double root i. In paper [1] fundamental systems of solutions (f.s.s.) for the equation (1) are constructed at great values of $|\lambda|$ containing fractional degree of the parameter both in the exponent and in the factor at the exponent. In paper [2] boundary value problem (1)-(2) without parameter in boundary conditions is investigated and series expension in eigen functions are obtained. In addition the Green's function of the problem behaves as $O\left(\lambda^{-\frac{1}{2}}\right)$ and if we take $p(x), q(x) \neq 0$, then giving boundary conditions $y(0) = 0, y(1) = 0$, we easily establish that the problem has eigen values.

On the contrary, for instance at $r(x) = 0$ under the same boundary conditions we can easily verify that the problem has no eigen values. This case is connected with the fact that f.s.s. of the equation (1) are defined not only by the leading coefficients of the polynomial on λ at the derivatives, but also by the properties of the low degree coefficients of the same polynomial.

Therefore in the case of f.s.s. boundary value problems containing only entire degree of the parameter need special consideration.

It is easy to establish the theorem from the results of paper [1].

Theorem. *Let* $p(x) = q(x) = 0,\ r(x) \in C^3[0,1]$. *Then, the equation (1) at upper and lower half-planes has f.s.s. admitting the representation*

$$y_i(x,\lambda) = \left[\eta_{i0}(x) + \frac{1}{\lambda}\eta_{i1}(x) + \frac{E_{ii}(x,\lambda)}{\lambda^2}\right] e^{i\lambda x}, \quad i = \overline{1,2}, \tag{3}$$

where $\eta_{i0}(x) = g_{1i}^{(0)}(x)$ *are fundamental systems of solutions of the second order differential equation,* $\eta_{i1}(x) = g_{1i}^{(1)}(x) + g_{2i}^{(1)}(x)$ *are the solutions of the second order in homogeneous differential equation, the function* $E(x,\lambda)$ *is bounded under sufficiently great values of* $|\lambda|$ *and continuous on* $x \in [0,1]$.

The solution of the equation (1) with in homogeneous right hand side $f(x)$ satisfying the boundary conditions (2) is determined by the formula

$$y(x,\lambda)=\int_0^1\frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}f(\xi)d\xi\,, \tag{4}$$

where

$$D(x,\xi,\lambda)=\begin{vmatrix} g(x,\xi,\lambda) & y_1(x,\lambda) & y_2(x,\lambda)\\ U_1(g)_x & \ldots & \ldots \\ U_2(g)_x & \vdots & \Delta(\lambda)\end{vmatrix},\ g(x,\xi,\lambda)=\pm\frac{\sum\limits_{k=1}^{2}y_k(x,\lambda)W_{2k}(\xi,\lambda)}{2W(\xi,\lambda)},\ \begin{matrix}+\xi\le x\\ -\xi\ge x\end{matrix}$$

The eigen-values of the problem (1-(2) are determined by the zeros of the characteristic determinant $\Delta(\lambda)$:

$$\Delta(\lambda)=\begin{vmatrix}\alpha_{11}(\lambda) & \alpha_{12}(\lambda)\\ \alpha_{21}(\lambda) & \alpha_{22}(\lambda)\end{vmatrix}\times\begin{vmatrix}\eta_{10}(0) & \eta_{20}(0)\\ \frac{d}{dx}\eta_{10}(0) & \frac{d}{dx}\eta_{20}(0)\end{vmatrix}+\left\{\begin{vmatrix}\alpha_{11}(\lambda) & \beta_{12}(\lambda)\\ \alpha_{21}(\lambda) & \beta_{22}(\lambda)\end{vmatrix}\times\begin{vmatrix}\eta_{10}(0) & \eta_{20}(0)\\ \frac{d}{dx}\eta_{10}(1) & \frac{d}{dx}\eta_{20}(1)\end{vmatrix}-\right.$$

$$-\begin{vmatrix}\alpha_{12}(\lambda) & \beta_{11}(\lambda)\\ \alpha_{22}(\lambda) & \beta_{21}(\lambda)\end{vmatrix}\times\begin{vmatrix}\eta_{10}(1) & \eta_{20}(1)\\ \frac{d}{dx}\eta_{10}(0) & \frac{d}{dx}\eta_{20}(0)\end{vmatrix}+\begin{vmatrix}\alpha_{12}(\lambda) & \beta_{12}(\lambda)\\ \alpha_{22}(\lambda) & \beta_{22}(\lambda)\end{vmatrix}\times\begin{vmatrix}\frac{d}{dx}\eta_{10}(0) & \frac{d}{dx}\eta_{20}(0)\\ \frac{d}{dx}\eta_{10}(1) & \frac{d}{dx}\eta_{20}(1)\end{vmatrix}-$$

$$\left.-\begin{vmatrix}\alpha_{11}(\lambda) & \beta_{11}(\lambda)\\ \alpha_{21}(\lambda) & \beta_{21}(\lambda)\end{vmatrix}\times\begin{vmatrix}\eta_{10}(1) & \eta_{20}(1)\\ \eta_{10}(0) & \eta_{20}(0)\end{vmatrix}\right\}e^{i\lambda}+\left\{\begin{vmatrix}\alpha_{11}(\lambda) & \beta_{12}(\lambda)\\ \alpha_{21}(\lambda) & \beta_{22}(\lambda)\end{vmatrix}\times\begin{vmatrix}\eta_{10}(0) & \eta_{20}(0)\\ \eta_{10}(1) & \eta_{20}(1)\end{vmatrix}-\right.$$

$$-\begin{vmatrix}\alpha_{12}(\lambda) & \beta_{11}(\lambda)\\ \alpha_{22}(\lambda) & \beta_{21}(\lambda)\end{vmatrix}\times\begin{vmatrix}\eta_{10}(1) & \eta_{20}(1)\\ \eta_{10}(0) & \eta_{20}(0)\end{vmatrix}-\begin{vmatrix}\alpha_{12}(\lambda) & \beta_{12}(\lambda)\\ \alpha_{22}(\lambda) & \beta_{22}(\lambda)\end{vmatrix}\times\begin{vmatrix}\eta_{10}(1) & \eta_{20}(1)\\ \frac{d}{dx}\eta_{10}(0) & \frac{d}{dx}\eta_{20}(0)\end{vmatrix}+$$

$$\left.+\begin{vmatrix}\alpha_{12}(\lambda) & \beta_{12}(\lambda)\\ \alpha_{22}(\lambda) & \beta_{22}(\lambda)\end{vmatrix}\times\begin{vmatrix}\eta_{10}(0) & \eta_{20}(0)\\ \frac{d}{dx}\eta_{10}(1) & \frac{d}{dx}\eta_{20}(1)\end{vmatrix}\right\}i\lambda e^{i\lambda}-\begin{vmatrix}\alpha_{12}(\lambda) & \beta_{12}(\lambda)\\ \alpha_{22}(\lambda) & \beta_{22}(\lambda)\end{vmatrix}\times\begin{vmatrix}\eta_{10}(0) & \eta_{20}(0)\\ \eta_{10}(1) & \eta_{20}(1)\end{vmatrix}\lambda^2e^{i\lambda}+$$

$$-\begin{vmatrix}\beta_{11}(\lambda) & \beta_{12}(\lambda)\\ \beta_{21}(\lambda) & \beta_{22}(\lambda)\end{vmatrix}\times\begin{vmatrix}\eta_{10}(1) & \eta_{20}(1)\\ \frac{d}{dx}\eta_{10}(1) & \frac{d}{dx}\eta_{20}(1)\end{vmatrix}e^{2i\lambda}=$$

$$=A_1(\lambda)+A_2(\lambda)e^{i\lambda}+A_3(\lambda)i\lambda e^{i\lambda}+A_4(\lambda)\lambda^2e^{i\lambda}+A_5(l)e^{2i\lambda}\,. \tag{5}$$

Assume that $A_k(\lambda)=\lambda^{H_k}\left[A_k+O\left(\frac{1}{\lambda}\right)\right],\ k=1{,}5;\lambda^{k-2}A_k=\lambda^{H_k}\left[A_k+O\left(\frac{1}{\lambda}\right)\right],$ $k=2,3,4$, $\alpha_{ij}(\lambda)=\sum\limits_{k=0}^{s_{ij}}\alpha_{ij}^{(k)}\lambda^k$, $\beta_{ij}(\lambda)=\sum\limits_{k=0}^{m_{ij}}\beta_{ij}^{(k)}\lambda^k$, $i,j=\overline{1,2}$.

Consider the case a) $s_{ij}=m_{ij}$, $i,j=\overline{1,2}$. Then for $\alpha_{11}^{(s_{11})}\cdot\alpha_{22}^{(s_{22})}\neq\alpha_{21}^{(s_{21})}\cdot\alpha_{12}^{(s_{12})}$, $\beta_{11}^{(m_{11})}\cdot\beta_{22}^{(m_{22})}\neq\beta_{21}^{(m_{21})}\cdot\beta_{12}^{(m_{12})}$ we obtain that $H_1=H_5=2(s_{11}+m_{11})$, the leading terms of polynomials $\lambda^{k-2}A_k$, $k=2,3,4$ respectively have the form:

$$A_2(\lambda):\lambda^{2s_{11}}\left\{\begin{vmatrix}\alpha_{11}^{(s_{11})} & \beta_{12}^{(m_{12})}\\ \alpha_{21}^{(s_{21})} & \beta_{22}^{(m_{22})}\end{vmatrix}+\begin{vmatrix}\alpha_{22}^{(s_{22})} & \beta_{21}^{(m_{21})}\\ \alpha_{12}^{(s_{12})} & \beta_{11}^{(m_{11})}\end{vmatrix}+\begin{vmatrix}\alpha_{11}^{(s_{11})} & \beta_{11}^{(m_{11})}\\ \alpha_{21}^{(s_{21})} & \beta_{21}^{(m_{21})}\end{vmatrix}\right\},$$

$$\lambda A_3(\lambda):\lambda^{2s_{11}+1}\left\{\begin{vmatrix}\alpha_{11}^{(s_{11})} & \beta_{11}^{(m_{11})}\\ \alpha_{22}^{(s_{21})} & \beta_{22}^{(m_{22})}\end{vmatrix}+\begin{vmatrix}\alpha_{12}^{(s_{12})} & \beta_{12}^{(m_{12})}\\ \alpha_{21}^{(s_{21})} & \beta_{21}^{(m_{21})}\end{vmatrix}\right\},$$

:

$$\lambda^2 A_4(\lambda): \lambda^{2s_{11}+2}\begin{vmatrix}\alpha_{11}^{(s_{22})} & \beta_{22}^{(m_{22})}\\ \alpha_{12}^{(s_{12})} & \beta_{12}^{(m_{12})}\end{vmatrix} + \lambda^{2s_{11}+1}\left\{\begin{vmatrix}\alpha_{22}^{s_{22}-1} & \beta_{12}^{m_{22}-1}\\ \alpha_{12}^{s_{12}} & \beta_{12}^{m_{12}}\end{vmatrix} + \begin{vmatrix}\alpha_{22}^{s_{22}} & \beta_{12}^{m_{22}}\\ \alpha_{12}^{s_{12}-1} & \beta_{12}^{m_{12}-1}\end{vmatrix}\right\}.$$

Thus: in the case a), in order $H_1 = H_5$ and the problem (1)-(2) to be regular, the fulfillment of the following conditions are sufficient

$$\begin{vmatrix}\alpha_{11}^{(s_{11})} & \alpha_{12}^{(s_{12})}\\ \alpha_{21}^{(s_{21})} & \alpha_{22}^{(s_{22})}\end{vmatrix} \neq 0, \quad \begin{vmatrix}\beta_{11}^{(m_{11})} & \beta_{12}^{(m_{12})}\\ \beta_{21}^{(m_{21})} & \beta_{22}^{(m_{22})}\end{vmatrix} \neq 0, \quad \begin{vmatrix}\alpha_{11}^{(s_{11})} & \beta_{11}^{(m_{11})}\\ \alpha_{22}^{(s_{22})} & \beta_{22}^{(m_{22})}\end{vmatrix} + \begin{vmatrix}\alpha_{12}^{(s_{12})} & \beta_{12}^{(m_{12})}\\ \alpha_{21}^{(s_{21})} & \beta_{21}^{(m_{21})}\end{vmatrix} = 0,$$

$$\begin{vmatrix}\alpha_{11}^{(s_{22})} & \beta_{22}^{(m_{22})}\\ \alpha_{12}^{(s_{12})} & \beta_{12}^{(m_{12})}\end{vmatrix} = 0, \quad \begin{vmatrix}\alpha_{22}^{(s_{22}-1)} & \beta_{22}^{(m_{22}-1)}\\ \alpha_{12}^{(s_{12})} & \beta_{12}^{(m_{12})}\end{vmatrix} + \begin{vmatrix}\alpha_{22}^{(s_{22})} & \beta_{22}^{(m_{22})}\\ \alpha_{12}^{(s_{12}-1)} & \beta_{12}^{(m_{12}-1)}\end{vmatrix} = 0. \tag{6}$$

Case b): $s_{ij} \neq m_{ij}, i, j = \overline{1,2}$, $s_{11} + s_{22} \neq s_{21} + s_{12}$. Then $H_1 = \max\{s_{11} + s_{22}, s_{21} + s_{12}\}$, $H_5 = \max\{m_{11} + m_{22}, m_{21} + m_{12}\}$; if $s_{11} + s_{22} = s_{21} + s_{12}$ we assume that $\alpha_{11}^{(s_{11})}\alpha_{22}^{(s_{22})} \neq \alpha_{21}^{(s_{21})}\alpha_{12}^{(s_{12})}$. If $m_{11} + m_{22} = m_{21} + m_{12}$ - then we assume that $\beta_{11}^{m_{11}}\beta_{22}^{m_{22}} \neq \beta_{21}^{m_{21}}\beta_{12}^{m_{12}}$.

The problem on the asymptotics of eigen values is reduced to the problem on the asymptotics of zeros of a quasi-polynomial. The problem on the distribution of roots of the latter has been studied well enough (see [1], [3]). Therefore we don't cite it here to avoid repetition.

Wronskii's determinant from f.s.s. has the representation

$$W(\xi,\lambda) = \left\{\begin{vmatrix}\eta_{10}(\xi) & \eta_{20}(\xi)\\ \eta_{111}(\xi) & \eta_{211}(\xi)\end{vmatrix} + \begin{vmatrix}\eta_{11}(\xi) & \eta_{21}(\xi)\\ \eta_{200}(\xi) & \eta_{200}(\xi)\end{vmatrix} + O\left(\frac{1}{\lambda}\right)\right\}\lambda e^{2i\lambda\xi}, \tag{7}$$

where $\eta_{k00}(\xi) = ig_{1k}^{(0)}(\xi)$, $\eta_{k11}(x) = \dfrac{d}{dx}g_{1k}^{(0)}(x) + i\left(g_{1k}^{(1)}(x) + g_{2k}^{(1)}(x)\right)$, $k = \overline{1,2}$.

To obtain the asymptotic representation of Green's function at the low half-plane we transform the determinant $\Delta(x,\xi,\lambda)$ so that the elements of the first column $g_0(x,\xi,\lambda)$, $g_1(\xi,\lambda)$, $g_2(\xi,\lambda)$ of the transformed determinant $\Delta_0(x,\xi,\lambda)$ do not increase. The elements of the column $\Delta(x,\xi,\lambda)$ with numbers 2, 3 multiply by $-\dfrac{W_{21}(\xi,\lambda)}{2W(\xi,\lambda)}, -\dfrac{W_{22}(\xi,\lambda)}{2W(\xi,\lambda)}$ respectively, and add together with corresponding elements of the first column and further expanding $\Delta_0(x,\xi,\lambda)$ in elements of the first row we have

$$\frac{\Delta_0(x,\xi,\lambda)}{\Delta(\lambda)} = g_0(x,\xi,\lambda) - \left[\eta_{10}(x) + \frac{1}{\lambda}\eta_{11}(x) + O\left(\frac{1}{\lambda^2}\right)\right]e^{i\lambda x}\left[g_1(\xi,\lambda)\cdot\frac{\Delta_{11}(\lambda)}{\Delta(\lambda)} - \right.$$

$$\left. - g_2(\xi,\lambda)\cdot\frac{\Delta_{21}(\lambda)}{\Delta(\lambda)}\right] + \left[\eta_{20}(x) + \frac{1}{\lambda}\eta_{21}(x) + O\left(\frac{1}{\lambda^2}\right)\right]e^{i\lambda x}\left[g_1(\xi,\lambda)\frac{\Delta_{12}(\lambda)}{\Delta(\lambda)} - g_2(\xi,\lambda)\frac{\Delta_{22}(\lambda)}{\Delta(\lambda)}\right]$$

In the determinant $\Delta_{\nu k}$, $\nu, k = \overline{1,2}$ of the exponential function having the greatest real part is the function $\exp(i\lambda)$. If we multiply the numerator and denominator $\Delta_0(x,\xi,\lambda)/\Delta(\lambda)$ by $\lambda^{-H_1}\exp(-2i\lambda)$ and remain that in all summands the real parts of the exponential function were non-positive at the lower half-plane, then $\Delta(\lambda)$ is lower bounded by positive constants exterior to small vicinities of eigen values, if we reject from the low half-plane the interiors of small circles $\overline{K_\delta}$ with centers in zeros $\Delta(\lambda)$ we get the formula: $G(x,\xi,\lambda)=O(1), |\lambda|\to\infty$. This estimate is valid and in the upper half-plane.

Examples:

1) The boundary conditions $y^{(k)}(0)+y^{(k)}(1)=0,\ k=0,1$ are regular.

2) $y(0)+y(1)=0, 3y(0)+y'(0)+2y(1)=0$. For such boundary conditions $\Delta(\lambda)=1-i\lambda e^{i\lambda}$, $\frac{\Delta_0(x,\xi,\lambda)}{\Delta(\lambda)}=g_0(x,\xi,\lambda)-\frac{1}{\Delta(\lambda)}e^{i\lambda(x-\xi)}\{\xi+(1-\xi-i\lambda\xi)e^{i\lambda}\}+$ $+\frac{xe^{i\lambda(x-\xi)}}{\Delta(\lambda)}\{1+(1-\xi-\xi i\lambda)e^{i\lambda}\}$, $g_0(x,\xi,\lambda)=\begin{cases}0, \xi\le x\\(\xi-x)e^{i\lambda(x-\xi)}, \xi\ge x\end{cases}$ and the Green's function increases in the low half-plane, as $\exp\{i\lambda x\}$.

3) $\beta_{i1}(\lambda)=0, \beta_{1i}(\lambda)\ne 0, i=\overline{1,2}$. Then in $\Delta(\lambda)$ the coefficient for $\exp\{2i\lambda\}$ equals to zero and multiplying the numerator and denominator of $\frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}$ by $\lambda^{-H}e^{-i\lambda}$ for the transformed Green's function we get the increasing function $\exp\{i\lambda x\}$, $\operatorname{Re} i\lambda\ge 0$.

4) $\beta_{1i}(\lambda)=0, \beta_{2i}(\lambda)=0, i=\overline{1,2}$. Then $\Delta(\lambda)$ has no denumerable number of zeros, the zeros are finite and the Green's function exponentially increases.

If the growth order of $A_3(\lambda), A_4(\lambda)\lambda^2$ is greater than H_1 or H_5, then we get that $G(x,\xi,\lambda)$ increases as $\lambda^\alpha, \alpha\ge 1$.

Theorem. *Let the boundary value problem (1)-(2) be regular and let the function* $\Phi_k(x)\in C^{4-k}[0,1]$, $\Phi_k^{(\nu)}(0)=\Phi_k^{(\nu)}(1)=0$, $k=\overline{0,3}$. *Then the formula of double expansion holds*

$$-\frac{1}{2\sqrt{-1}}\sum_\nu\int_{R_\nu}\lambda^s d\lambda\int_0^1 G(x,\xi,\lambda)[\lambda\Phi_0(\xi)+\Phi_1(\xi)]d\xi=\begin{cases}\Phi_0(x), s=0\\ \Phi_1(x), s=1\end{cases}, \tag{8}$$

where R_ν *are small circles with a center in zeros* $\Delta(\lambda)$, *which uniformly converges at all* $x\in[0,1]$.

References

[1]. Тамаркин Я.Д. *О некоторых общих задачах теории обыкновенных дифференциальных уравнений и разложений произвольных функций в ряды.* ПГ, 1917.

[2]. Печенцов А.С. *Краевые задачи для дифференциальных уравнений, содержащих параметр, с кратными корнями характеристического уравнения.* Дифференциальные уравнения, 1984, №2, с. 263-273.

[3]. Садовничий В.А., Любишкин В.Л., Белаббаси Ю. *О нулях целых функций одного класса.* В кн.: Труды семинара им. Г. Петровского, вып.8, М., МГУ, 1982, с. 211-217.

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