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ON THE SPECTRUM OF A CLASS OF NONSELF-ADJOINT DIFFERENTIAL SINGULAR OPERATORS

Abstract

A spectrum of a class of nonself-adjoint differential operators of $2n$ *order, determined on the whole axis with coefficients not being infinite small and polynomially dependent on a complex spectral parameter is given in the paper.*

Earlier we have considered the cases [1], [2] when the continuous part of the spectrum of nonself-adjoint differential singular high order operators was on the rays of a complex λ-plane. In the suggested wider class of nonself-adjoint differential singular operators not only the presence of the continuous spectrum on the curves in a complex λ-plane is proved, but also the remaining constituents of the spectrum of the family are characterized.

Consider a differential family of operators:

$$L(\lambda)=L_0+L_1(\lambda), \tag{1}$$

where L_0 and $L_1(\lambda)$ are generated by differential expressions of the form

$$L_0=\sum_{i=0}^{2n}q_i\frac{d^{2n-i}}{dx^{2n-i}}\ ,\quad q_0\equiv 1\ ;$$

$$L_1=\sum_{j=2}^{2n}p_j\frac{d^{2n-j}}{dx^{2n-j}}\ ;$$

$$P_j(x,\lambda)=\lambda^{j-1}p_{j1}(x)+\cdots+p_{jj}(x). \tag{2}$$

The domain of the operator $L(\lambda)$ is determined analogously [1], p.145.

Assume that $(x^r+1)q_j(x)\in L_1(-\infty,\infty)$, where r is the highest multiplicity of the root of the equation $p'(\mu)=0$, where

$$p(\mu)=\mu^{2n}+q_1\mu^{2n-1}+q_2\mu^{2n-2}+\cdots+q_{2n-1}\mu+q_{2n}. \tag{3}$$

Consider the equation

$$L(\lambda)y+\lambda^{2n}y=f\ . \tag{4}$$

Rewrite it in the form

$$y^{(2n)}+q_1y^{(2n-1)}+\sum_{j=2}^{2n-1}\left[q_j+p_j(x,\lambda)\right]y^{(2n-1)}+\left[q_{2n}+p_{2n}(x,\lambda)+\lambda^{2n}\right]y=f\ .$$

This equation is equivalent to the system of $2n$ equations of the first order

$$Y'=(Q+P)Y+F\ , \tag{5}$$

where

$$Y=\begin{pmatrix}y_1\\ \vdots\\ y_{2n}\end{pmatrix};\quad Y'=\begin{pmatrix}y_n'\\ \vdots\\ y_{2n}'\end{pmatrix};\quad F=\begin{pmatrix}0\\ \vdots\\ f\end{pmatrix};$$

$$P=\left(p_{jk}\right)=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -P_{2n}(x,\lambda)-\lambda^{2n} & -P_{2n-1}(x,\lambda) & -P_{2n-2}(x,\lambda) & \cdots & -P_1(x,\lambda) \end{pmatrix},\ P_1\equiv 0;$$

$$Q=\left(q_{jk}\right)=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -q_{2n} & -q_{2n-1} & -q_{2n-2} & \cdots & -q_2 & -q_1 \end{pmatrix}.$$

To obtain the Green function for (4), first of all we construct some solutions of (5) for $F=0$. Note that if $p(\mu)=\lambda^{2n}$ has $2n$ different solutions $\mu_1,\dots,\mu_{2n}$, then $Z'=QZ$ has a fundamental matrix $M\exp[\theta x]$, where $\theta=\lfloor\mu_j\delta_{jk}\rfloor$ and $M=\left|(\mu_j)^{k-1}\right|$.

If in (5) $F=0$, we can write the integral equation equivalent to (5) in the form:

$$Y(x,\lambda)=M\exp[\theta x]C_0+\int^x M\exp[\theta(x-\xi)]M^{-1}P(\xi;\lambda)Y(\xi,\lambda)d\xi\,, \qquad (6)$$

where $C_0=const$ is $2n\times 1$ matrix, and the lower bounds of integrals (of each element in the matrix column) are arbitrary. The solutions (6) will be asymptotic to the solutions of the equation $Z'=QZ$.

We solve (6) in domains D that are simply connected and do not contain circles γ_{jk} inside themselves. The circles γ_{jk} are determined by the equations

$$\operatorname{Im}\mu_j=\operatorname{Im}\mu_k\,.$$

Let for some $\varepsilon>0$ the functions $p_{ki}(x)$ from (2) satisfy the condition

$$\left|p_{ki}(x)\right|\le ce^{-\varepsilon x}, \qquad i\le k\,. \qquad (7)$$

Theorem 1. *If (7) holds, then there are solutions $\varphi_1,\dots,\varphi_{2n}$ and $\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_{2n}$ of equation (6), that exist for all bounded λ in $\overline{D}$. The matrices*

$$\Phi=\left[\varphi_1,\dots,\varphi_{2n}\right]$$

and

$$\widetilde{\Phi}=\left[\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_{2n}\right]$$

are holomorphic in $\lambda\left(\lambda\in\overline{D}\right)$ for fixed x and have the following asymptotic behavior:

$$\begin{aligned}\Phi(x,\lambda)&=M\exp[i\theta\, x](I+0(1)),\ x\to+\infty\,,\\ \widetilde{\Phi}(x,\lambda)&=M\exp[i\theta\, x](I+0(1)),\ x\to-\infty,\end{aligned} \qquad (8)$$

where $I=\lfloor\delta_{jk}\rfloor$ is a constant matrix.

This theorem is proved in a complete analogy to theorem 8.1 (see [3], p.104).

We shall also be interested in asymptotic behavior of $\Phi(x,\lambda)$ and $\widetilde{\Phi}(x,\lambda)$ for $|\lambda|\to\infty$. For large $|\lambda|$, μ_i may be enumerated as

$$\mu_j=\alpha_j\lambda\left(1+O(|\lambda|)^{-1}\right),$$

where $0\le\arg\lambda\le\dfrac{\pi}{n}$ and $\alpha_j=\exp\dfrac{\pi ij}{n}$.

Then it is easy to count that

$$\Phi(x,\lambda)=M\exp[i\theta x]\exp\left[-\frac{1}{2n}\int_0^x\beta(t)dt\right]\left[1+O\left(|\lambda|^{-1}\right)\right];\ \ |\lambda|\to\infty, \tag{9}$$

$$\beta=[\beta_k\delta_{ki}],\ \ \beta_k(x)=\mu_k p_{2n,1}(x)+\mu_k^2 p_{2n-1,1}(x)+\cdots+\mu_k^{2n-1}p_{2,1}(x).$$

The operator $L(\lambda)$ is a closed operator on $L^p(-\infty,\infty)$. We consider the case when $p(\mu)=\lambda^{2n}$ has no real values. Then we can enumerate the solution $\mu_1,...,\mu_{2n}$ such that

$$\operatorname{Im}\mu_1\ge\operatorname{Im}\mu_2\ge\cdots\ge\operatorname{Im}\mu_m>0>\operatorname{Im}\mu_{m+1}\ge\cdots\ge\operatorname{Im}\mu_{2n}.$$

Decompose our matrices

$$\Phi=\begin{bmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{bmatrix},\qquad \widetilde{\Phi}=\begin{bmatrix}\widetilde{\Phi}_{11} & \widetilde{\Phi}_{12}\\ \widetilde{\Phi}_{21} & \widetilde{\Phi}_{22}\end{bmatrix},$$

where each of Φ_{11} and $\widetilde{\Phi}_{11}$ has the dimension of $m\times m$.

Define the matrix

$$\Psi=\begin{bmatrix}\Phi_{11} & \widetilde{\Phi}_{12}\\ \Phi_{21} & \widetilde{\Phi}_{22}\end{bmatrix}$$

under the condition that Ψ^{-1} exists. And, finally, define the matrix

$$W_j(\lambda)=\pm(\det\Psi)\exp(iq_1x). \tag{10}$$

Introduce the following notations: let a resolvent set, a spectrum, a point spectrum, a remainder spectrum, and a continuous spectrum of the operator $L(\lambda)$ be denoted by $\pi(L)$, $\sigma(L)$, $P\sigma(L)$, $R\sigma(L)$ and $C\sigma(L)$ respectively.

Then the following main theorem holds.

Theorem 2. *If (7) holds and $p(\mu)=\lambda^{2n}$ has no real solutions, then $\lambda\in\rho(L)$ or $\lambda\in P\sigma(L)$, moreover $\lambda\in P\sigma(L)$ if and only if $W_j(\lambda)=0$ for $\operatorname{Im}\mu_j\ne\operatorname{Im}\mu_k$. The curve $\lambda^{2n}=p(t)$ is contained in $\sigma(L)$ and it contains $C\sigma(L)$ and $R\sigma(L)$, and the points of $P\sigma(L)$ and of $R\sigma(L)$, lying on it, form nowhere dense set on any arc that does not lie between two D_j, in which $W_j(\lambda)\equiv 0$. Each part of $\sigma(L)$ is independent on p, excepting the case $p=\infty$, where $\sigma(L)$ is all $P\sigma(L)$ excluding possible the points lying on $\lambda^{2n}=p(t)$ for which $\operatorname{Im}\mu_j\ne\operatorname{Im}\mu_k$.*

Prove this. If $p(\mu)=\lambda^{2n}$ has no real solutions, then either $\lambda\in\rho(L)$ or $\varphi_1,...,\varphi_m,\widetilde{\varphi}_{m+1},...,\widetilde{\varphi}_{2n}$ are dependent solutions. In this case there are constants c_j that not all are equal to zero, such that

$$\Psi\equiv\sum_{j=1}^{m}c_j\varphi_j+\sum_{j=m+1}^{2n}c_j\widetilde{\varphi}_j.$$

If χ is the first component of Ψ, obviously that $\chi\in L^p$ for all $p\ge 1$ and $\left(L-\lambda^{2n}\right)\chi=0$, therefore $\lambda\in P\sigma(L)$.

On the arc $\lambda^{2n}=p(t)$ we have $W_j(\lambda)$ on one hand and $W_k(\lambda)$ on the other hand. If both are identically equal to zero, then all the points of the arc are on the closure of $P\sigma(L)$ and consequently are in $\sigma(L)$. If $W_j(\lambda)\not\equiv 0$, then at any point on $\lambda^{2n}=p(t)$, where it is not equal to zero, solution (3), where $f(x)=0$ for $|x|\ge a$, that belongs to L^p

is the first component of $\int_{-a}^{a} K(x;\xi;\lambda)F(\xi)d\xi$, where $F(\xi)$ is a vector-column with $f(\xi)$ the last term and with all remained zeros. If $\left(L(\lambda)-\lambda^{2n}\right)y=0$ has a solution in L^p, then either

$$\varphi_m(x)\sim Ce^{itx} \quad \text{for} \quad x\to\infty$$

or

$$\tilde{\varphi}_{m+1}(x)\sim Ce^{itx} \quad \text{for} \quad x\to-\infty .$$

Thus, in order for this solution to belong to L^p $(p\neq\infty)$, we must have the m-th or $(m+1)$-th term in $\int_{-a}^{a}\Psi_j^{-1}(\xi)F(\xi)d\xi$ equal to zero. We can easily choose F so that this not be fulfilled, therefore $\left(L(\lambda)-\lambda^{2n}\right)^{-1}$ may not be determined in these points and $\lambda\in\sigma(L)$.

For $\lambda_0^{2n}=p(t_0)$ to belong to $P\sigma(L)$ for $p=\infty$, we must have linear dependence between the solutions that are exponentially small in $+\infty$ and those that are exponentially small in $-\infty$. This means that $W_j(\lambda)$ and $W_k(\lambda)$ both must be zeros and consequently, these points may not be dense on the arc $\lambda^{2n}=p(t)$, if only $W_j(\lambda)\not\equiv 0$ and $W_k(\lambda)\not\equiv 0$.

Points $R\sigma(L)$ may not be dense on the arc if only these $P\sigma(L^*)$ are dense on the corresponding arc $\lambda^{2n}=p^*(t)$, and it means that $W_j^*(\lambda)\equiv 0$ and $W_k^*(\lambda)\equiv 0$. Thus, all the points of this arc of the curve $\lambda^{2n}=p^*(t)$ are conjugated points in $P\sigma(L^*)$ and therefore they belong to $\sigma(L)$. On above said, they belong to $P\sigma(L)$ and $W_j(\lambda)$ and $W_k(\lambda)$ must be identical zeros. Therefore, $R\sigma(L)$ may not be dense on the arc of the curve $\lambda^{2n}=p(t)$, that doesn't lye between two D_j, in which $W_j(\lambda)\equiv 0$.

References

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