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PROPERTIES OF SOLUTIONS OF THE CONJUGATE PROBLEM FOR THE PARABOLIC EQUATIONS

Abstract

The simple mathematical model of friction welding process is described by conjugate problem for the parabolic equations. In this article, it is investigating some properties of the solutions of this problem.

In article [1] there was proved the theorem on the solvability of following conjugate problem:

$$-\frac{\partial u}{\partial t}+k\frac{\partial^2 u}{\partial x^2}=0, \quad x\in(-l,0)\cup(0,l),\ t\in(0,T),\ l,T>0 \quad , \tag{1}$$

$$u\big|_{t=0}=0, \tag{2}$$

$$\left.\begin{aligned} & u\big|_{x=0-}=u\big|_{x=0+}, \\ & \frac{\partial u}{\partial x}\bigg|_{x=0-}=\frac{\partial u}{\partial x}\bigg|_{x=0+}+w, \\ & \left[\frac{\partial u}{\partial x}-\alpha u\right]_{x=-l}=0,\ \left[\frac{\partial u}{\partial x}+\alpha u\right]_{x=+l}=0, \end{aligned}\right\} \tag{3}$$

where k, w, α are positive constants. Also the confirmation [1, theorem 3], that for any $t>0$ solution of the problem (1)-(3) satisfies the inequality was formulated there:

$$\frac{1}{\alpha}\int_{-l}^{+l}\left(\frac{\partial u}{\partial x}\right)^2 dx+u^2(t,-l)+u^2(t,l)\le\frac{w}{\alpha}u(t,0). \tag{4}$$

But the proof of this inequality contains in non-precisions, what about the author informed the editorial board of the journal [2].

In the present work some properties of solution of problems (1)-(3) are proved and validity of (4) is established.

Theorem 1. *For any t,x the solution of the problem (1)-(3) is non-negative, i.e.* $u(t,x)\ge 0$.

Proof. Let's assume the contrary. Let the solution is negative on some subset of set $(0,T)\times[(-l,0)\cup(0,l)]$. Let's introduce the functions

$$u_p(t,x)=\frac{1}{2}(|u|+u), \qquad u_m(t,x)=\frac{1}{2}(|u|-u).$$

It is obvious, that $u_p\cdot u_m\equiv 0, u_m\ge 0, u_p\ge 0$ and $u=u_p-u_m$. Let's multiply the equation (1) by the function u_m and integrate it to $(0,t)\times(-l,l)$

$$\int_{-l}^{l}\int_{0}^{t}\frac{\partial u}{\partial t}u_m dtdx=k\int_0^t\int_{-l}^{l}\frac{\partial^2 u}{\partial x^2}u_m dxdt .$$

Taking into account conditions (2), (3) we investigate the left and right sides of this equality

$$\int_{-l}^{l}\int_{0}^{t}\frac{\partial u}{\partial t}u_m dtdx=-\int_{-l}^{l}\int_{0}^{t}\frac{\partial u_m}{\partial t}u_m dtdx=-\int_{-l}^{l}\frac{u_m^2}{2}dx<0,$$

$$k\int_{0}^{t}\int_{-l}^{l}\frac{\partial^2 u}{\partial x^2}u_m dxdt=-k\int_{0}^{t}\int_{-l}^{l}\frac{\partial^2 u_m}{\partial x^2}u_m dxdt=$$

$$=k\int_{0}^{t}\left[\alpha u_m^2(t,-l)+\alpha u_m^2(t,l)+\int_{-l}^{+l}\left(\frac{\partial u_m}{\partial x}\right)^2 dx+wu_m(t,0)\right]dt\geq 0.$$

The obtained contradiction shows that our assumption is not right. The theorem has been proved.

So, for any $\tau>0$ $u(\tau,x)\geq 0$, moreover case $u(\tau,x)\equiv 0$ is exepted. Hence we have $\frac{\partial u}{\partial t}(0,x)\equiv\lim\limits_{\tau\to 0}\frac{u(\tau,x)}{\tau}\geq 0$.

Further we will use the fact, that the solution of equation (1) has derivatives with all orders on t for $x\in(-l,0)\cup(0,l)$ [3, p. 500].

Theorem 2. *For any t, x the solution of problem (1)-(3) satisfies the inequality:*

$$\frac{\partial u}{\partial x}(t,x)\geq 0. \tag{5}$$

Proof. Let's denote $z=\frac{\partial u}{\partial t}$. Differentiating equality $z=k\frac{\partial^2 u}{\partial x^2}$ by t, we obtain the equation:

$$-\frac{\partial z}{\partial t}+k\frac{\partial^2 z}{\partial x^2}=0. \tag{1'}$$

By analogy, we will obtain boundary conditions with respect to $z=z(t,x)$

$$z\Big|_{t=0}=\frac{\partial u}{\partial t}(0,x)\geq 0, \tag{2'}$$

$$\left.\begin{array}{l} z\big|_{x=0-}=z\big|_{x=0+}, \\ \left.\frac{\partial z}{\partial x}\right|_{x=0-}=\left.\frac{\partial z}{\partial x}\right|_{x=0+}, \\ \left[\frac{\partial z}{\partial x}-\alpha z\right]_{x=-l}=0,\ \left[\frac{\partial z}{\partial x}+\alpha z\right]_{x=+l}=0, \end{array}\right\} \tag{3'}$$

Now applying the scheme of the proof of theorem 1 to problem (1')-(3') we will obtain validity of inequality (5).

Theorem 3. *The inequality (4) is valid.*

Proof. By virtue of inequality (5) and theorem (1) for any t, x

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{k}\frac{\partial u}{\partial t}\geq 0 \quad\text{and}\quad u\cdot\frac{\partial^2 u}{\partial x^2}\geq 0.$$

Integrating the right-hand and left-hand sides of the last inequality in interval $(-l,l)$, we have

$$0 \le \int_{-l}^{+l} u \frac{\partial^2 u}{\partial x^2} dx = \left[u \frac{\partial u}{\partial x} \right]_{x=l} - \left[u \frac{\partial u}{\partial x} \right]_{x=0+} + \left[u \frac{\partial u}{\partial x} \right]_{x=0-} - \left[u \frac{\partial u}{\partial x} \right]_{x=-l} - \int_{-l}^{+l} \left(\frac{\partial u}{\partial x} \right)^2 dx =$$

$$= -\alpha u^2(t,l) - \alpha u^2(t,-l) + wu(t,0) - \int_{-l}^{+l} \left(\frac{\partial u}{\partial x} \right)^2 dx .$$

Hence we obtain inequality (4). The theorem has been proved.

References

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