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ON THE CONVERGENCE RATE OF TRUNCATED HYPERSINGULAR INTEGRALS IN CHARACTERISTIC POINTS AND IN THE NORM OF THE SPACE $L_p(R^n)$

Abstract

The convergence rate of truncated hypersingular integrals in characteristic points and in the norm of the space $L_p(R^n)$ is studied in the paper.

As we known Bessel and Riesz potentials are determined in Fourier transforms by the equalities (see, for example, [1], [2])

$$\left(J^\alpha \varphi\right)^\wedge(x) = (1+|x|^2)^{-\alpha/2} \hat{\varphi}(x),$$

$$(I^\alpha \varphi)^\wedge(x) = |x|^{-\alpha} \hat{\varphi}(x).$$

Potential reversing constructions are realized in the form of hypersingular integrals regarded as some limit of corresponding "truncated" integrals. In this paper, we study the relation between the "smoothness order" of the density $\varphi \in L_p(R^n)$ of potentials $J^\alpha \varphi$ and $I^\alpha \varphi$ and the convergence rate of truncated integrals.

Introduce a truncated integral by the following scheme. As we know, the Gauss-Weierstrass integral of the function $f(x), x \in R^n$, has the form

$$(Gf)(x,t) = \left(W(\cdot,t) * f\right)(x) = \int_{R^n} W(y,t) f(x-y) dy, t > 0,$$

where the Gauss-Weierstrass kernel $W(y,t)$ is determined by the formula

$$W(y,t) = (4\pi t)^{-n/2} \cdot \exp(-|y|^2 / 4t)$$

It is known that a family of operator $(Gf)(\cdot,t)$, $t > 0$ forms a semigroup with respect to a parameter (see, for example, [3], [4], [5]).

Introduce also a modified Gauss-Weierstrass semigroup:

$$(G^{[m]}f)(x,t) = e^{-t}(Gf)(x,t), \ t > 0,$$

where $(Gf)(x,t)$ is the above determined semigroup.

Let $\ell > \dfrac{\alpha}{2} + 1$ and

$$\Re(\alpha,\ell) = \int_0^\infty \frac{\left(1 - e^{-t}\right)^\ell}{t^{1+\alpha}} dt, \ (0 < \alpha < \ell, \ \ell \text{ is a natural number}).$$

Introduce the truncated integrals

$$D_\varepsilon^\alpha f = \frac{1}{\Re\left(\frac{\alpha}{2},\ell\right)} \cdot \int\limits_\varepsilon^\infty \left(\sum_{k=0}^\ell \binom{\ell}{k}(-1)^k \left(G^{[m]}f\right)(x,k\tau)\right)\frac{d\tau}{\tau^{1+\frac{\alpha}{2}}} \equiv$$

$$\equiv \frac{1}{\Re\left(\frac{\alpha}{2},\ell\right)} \cdot \int\limits_\varepsilon^\infty \Delta_\tau^\ell \left[\left(G^{[m]}f\right)(x,\cdot)\right](0) \cdot \frac{d\tau}{\tau^{1+\frac{\alpha}{2}}};$$

(1)

and

$$\widetilde{D}_\varepsilon^\alpha f = \frac{1}{\Re\left(\frac{\alpha}{2},\ell\right)} \cdot \int\limits_\varepsilon^\infty \left(\sum_{k=0}^\ell \binom{\ell}{k}(-1)^k (Gf)(x,k\tau)\right)\frac{d\tau}{\tau^{1+\frac{\alpha}{2}}} \equiv$$

$$\equiv \frac{1}{\Re\left(\frac{\alpha}{2},\ell\right)} \cdot \int\limits_\varepsilon^\infty \Delta_\tau^\ell \left[(Gf)(x,\cdot)\right](0) \cdot \frac{d\tau}{\tau^{1+\frac{\alpha}{2}}}.$$

(2)

Bessel's $J^\alpha\varphi$ and Riesz's $I^\alpha\varphi$ potentials are determined by the formulas

$$\left(J^\alpha\varphi\right)^\wedge(x) = (1+|x|^2)^{-\alpha/2}\,\hat{\varphi}(x),$$

$$(I^\alpha\varphi)^\wedge(x) = |x|^{-\alpha}\,\hat{\varphi}(x),$$

where $^\wedge$ means the Forier transformation in R^n.

Constructions (1)-(2) and above-determined operators semigroup has a close relation that is described as follows:

a) if $\varphi \in L_p(R^n)$, $1 \le p \le \infty$ and $0 < \alpha < \infty$,

then for any $\varepsilon > 0$

$$\left(D_\varepsilon^\alpha J^\alpha\varphi\right)(x) = \int\limits_0^\infty K_{\frac{\alpha}{2}}^{(\ell)}(\eta) \cdot \left(G^{[m]}\varphi\right)(x,\varepsilon\eta)d\eta,$$

(3)

b) if $\varphi \in L_p(R^n)$, $1 \le p < \infty$ and $0 < \alpha < \frac{n}{p}$, then for any $\varepsilon > 0$

$$\left(\widetilde{D}_\varepsilon^\alpha I^\alpha\varphi\right)(x) = \int\limits_0^\infty K_{\frac{\alpha}{2}}^{(\ell)}(\eta) \cdot (G\varphi)(x,\varepsilon\eta)d\eta.$$

(4)

In these formulas $K_\nu^{(\ell)}(\eta)$ is McDonald's modified function.

The main result is

Theorem 1. *Let* $\varphi \in L_p(R^n)$, $1 \le p \le \infty$ *and at the point* $x^0 \in R^n$ *it is fulfilled the condition*

$$\sup_{0<r<\delta} \frac{1}{r^{n+\beta}} \cdot \iint\limits_{|z|<r} |\varphi(x^0 - z) - \varphi(x^0)|dz = A_\delta(x^0) < \infty \quad.$$

(Λ_0)

Then for $\varepsilon \to 0$

$$\left(D_\varepsilon^\alpha J^\alpha\varphi\right)(x^0) - \varphi(x^0) = 0(\varepsilon^\beta).$$

Proof. Taking integral representation (3) into account we can write

$$\left(D_\varepsilon^\alpha J^\alpha\varphi\right)(x^0) - \varphi(x^0) = \int\limits_0^\infty K_{\frac{\alpha}{2}}^{(\ell)}(\eta)\left[\left(G^{[m]}\varphi\right)(x^0,\varepsilon\eta) - \varphi(x^0)\right]d\eta.$$

Estimate the difference

$$\left|\left(G^{[m]}\varphi\right)\!\left(x^0,\varepsilon\eta\right)-\varphi(x^0)\right|=i_0$$

for fixed $\eta>0$.

We have

$$i_0=\left|e^{-\varepsilon\eta}(G\varphi)(x^0,\varepsilon\eta)-\varphi(x^0)\right|\le\left|e^{-\varepsilon\eta}(G\varphi)(x^0,\varepsilon\eta)-\right.$$
$$-\left.(G\varphi)(x^0,\varepsilon\eta)\right|+\left|(G\varphi)(x^0,\varepsilon\eta)-\varphi(x^0)\right|=i_1+i_2.$$

Estimate $i_1=\left|(G\varphi)(x^0,\varepsilon\eta)\right|\cdot(1-e^{-\varepsilon\eta})$.

It is easy to prove that $1-e^{-\varepsilon\eta}\le e\cdot\varepsilon\eta$.

So, $i_1\le c_1\cdot\varepsilon\eta$. Now estimate i_2.

$$i_2=\left|(G\varphi)(x^0,\varepsilon\eta)-\varphi(x^0)\right|=\left|\int\limits_{R^n}W(y,\varepsilon\eta)\varphi(x^0-y)dy-\varphi(x^0)\right|=$$

$$=\left|\int\limits_{R^n}W(y,\varepsilon\eta)\left[\varphi(x^0-y)-\varphi(x^0)\right]dy\right|=\left|\int\limits_{|y|\le\delta}...+\int\limits_{|y|>\delta}...\right|\le\left|\int\limits_{|y|<\delta}...\right|+\left|\int\limits_{|y|>\delta}...\right|=i_3+i_4.$$

Estimate i_3. For this introduce the function

$$g(x)=\begin{cases}\varphi(x)-\varphi(x^0),\text{ for }&|x-x_0|\le\delta\\0&,\text{ for }&|x-x_0|>\delta\end{cases}$$

We have

$$i_3\le c_2\cdot A_\delta(x^0)\cdot\int\limits_0^\infty\widetilde{W}(r,\varepsilon\eta)r^{n+\beta-1}dr,$$

where $\widetilde{W}(r,\varepsilon\eta)=W(y,\varepsilon\eta)\big|_{|y|=r}$.

Taking into account that

$$\int\limits_0^\infty\widetilde{W}(r,\varepsilon\eta)r^{n+\beta-1}dr=c_3\cdot\int\limits_0^\infty(\varepsilon\eta)^{-n}e^{-\frac{r^2}{(\varepsilon\eta)^2}}r^{n+\beta-1}dr=$$

$$=c_3\cdot(\varepsilon\eta)^{-n}\cdot\int\limits_0^\infty e^{-\tau^2}(\varepsilon\eta)^{n+\beta}\tau^{n-1+\beta}d\tau=c_4\cdot(\varepsilon\eta)^\beta,$$

we get $i_3\le c_5(\varepsilon\eta)^\beta$.

Now estimate i_4. We have

$$i_4\le\int\limits_{|y|>\delta}W(y,\varepsilon\eta)\cdot\left|\varphi(x^0-y)-\varphi(x^0)\right|dy\le$$

$$\le\left|\varphi(x^0)\right|\cdot\int\limits_{|y|>\delta}W(y,\varepsilon\eta)dy+\left\|\varphi\right\|_{L_p}\cdot\left\|W(y,\varepsilon\eta)\right\|_{L_{p'}(|y|>\delta)}=i_5+i_6;(1/p+1/p')=1.$$

Estimate i_5 and i_6.

In a similar way it is proved

Theorem 2. *Let* $\varphi\in L_p(R^n)$, $1\le p<\dfrac{n}{\alpha}$ *and at the point* $x^0\in R^n$ *the relation $(A_{()})$*

be satisfied. Then

$$\left(\widetilde{D}_1^\alpha J^\alpha\varphi\right)\!(x^0)-\varphi(x^0)=0(\varepsilon^\beta),\text{ for }\varepsilon\to0.$$

$$i_5 - \left|\varphi(x^0)\right| \cdot \int\limits_{|y|>\delta} W(y,\varepsilon\eta)dy - c_6 \cdot (\varepsilon\eta)^{-n} \cdot \int\limits_{\delta}^{\infty} r^{n-1} e^{-\left(\frac{r}{\varepsilon\eta}\right)^2} dr \le$$

$$\le c_7 \cdot e^{-\frac{\delta}{\varepsilon\eta}} \cdot \int\limits_0^{\infty} e^{-r} r^{n-1} dr = c_8 \cdot e^{\frac{\delta}{\varepsilon\eta}}$$

Taking into account that $\inf\limits_{\tau \ge 0}\left(e^{\frac{\delta}{\tau}} \cdot \tau\right) = e\delta$, from the last estimate we have

$i_5 \le (e\delta)^{-1} \cdot c_8 \cdot \varepsilon\eta = c_9 \cdot \varepsilon\eta$, where c_9 is independent of ε and η

$$i_6 = \left\|\varphi\right\|_{L_p} \cdot \left\|W(y,\varepsilon\eta)\right\|_{L_p,(|y|>\delta)} == c_{10} \cdot \left\{\int\limits_{\delta/\varepsilon\eta}^{\infty} (\varepsilon\eta)^{-np'} (\varepsilon\eta)^n \cdot r^{n-1} e^{-r^2 p'} dr\right\}^{1/p'} =$$

$$= c_{10} \cdot (\varepsilon\eta)^{-n\cdot\frac{p'-1}{p'}} \cdot \left(\int\limits_{\delta/\varepsilon\eta}^{\infty} r^{n-1} e^{-r^2 p'} dr\right)^{1/p'} \le c_{10} \cdot (\varepsilon\eta)^{-n\left(1-\frac{1}{p'}\right)} \cdot e^{-\frac{\delta}{\varepsilon\eta}} \cdot \left(\int\limits_0^{\infty} r^{n-1} e^{-rp'} dr\right)^{1/p'} =$$

$$= c_{11} \cdot e^{-\frac{\delta}{\varepsilon\eta}} \cdot (\varepsilon\eta)^{-\frac{n}{p}}.$$

Taking into account that $\inf\limits_{\tau \ge 0} \tau \cdot \tau^{\frac{n}{p}} \cdot e^{\frac{\delta}{\tau}} = \left(\frac{e\delta p}{n+p}\right)^{\frac{n+p}{p}}$, from the last estimate we have

$$i_6 \le c_{11} \cdot \left(\frac{n+p}{ep\delta}\right)^{\frac{n+p}{p}} \cdot \varepsilon\eta = c_{12} \cdot \varepsilon\eta,$$

where c_{12} is independent of ε and η.

Summing up all obtained estimates, we deduce that

$$\left|(G^{[m]}\varphi)(x^0,\varepsilon\eta) - \varphi(x^0)\right| \le c_{13} \cdot \varepsilon\eta + c_{14} \cdot (\varepsilon\eta)^{\beta}.$$

Thus, for small $\varepsilon > 0$

$$\left|\left(D_{1,\varepsilon}^{\alpha} J^{\alpha}\varphi\right)(x^0) - \varphi(x^0)\right| \le \int\limits_0^{\infty}\left|K_{\alpha/2}^{(\ell)}(\eta)\right| \cdot \left(c_{13} \cdot \varepsilon\eta + c_{14}(\varepsilon\eta)^{\beta}\right)d\eta \le$$

$$\le \varepsilon^{\beta} \cdot \int\limits_0^{\infty}\left|K_{\alpha/2}^{(\ell)}(\eta)\right| \cdot \left(c_{13} \cdot \eta + c_{14}\eta^{\beta}\right)d\eta.$$

Taking into account asymptotics for $K_{\alpha/2}^{(\ell)}(\eta)$ and the condition $\ell > \frac{\alpha}{2}+1$ we obtain that the integral at the right hand side of the last estimate converges and this completes the proof of the theorem.

Then we study the convergence rate of truncated hypersingular integrals (1) and (2).

Theorem 3. *Let the function* $\varphi \in L_p(R^n)$, $1 \le p \le \infty$ *satisfy the condition*

$$\sup\limits_{0<r<\delta} \frac{1}{r^{n+\beta}} \cdot \int\limits_{|z|<r}\left\|\varphi(x-z) - \varphi(x)\right\|_{L_p(R^n(x))}dz = A_\delta < \infty. \tag{A}$$

Then

$$\left\|D_{\varepsilon}^{\alpha} J^{\alpha}\varphi - \varphi\right\|_{L_p} = 0(\varepsilon^{\beta}), \text{ for } \varepsilon \to 0.$$

Proof. From the estimate

$$\left|\left(D_{1,\varepsilon}^{\alpha} J^{\alpha} \varphi\right)(x) - \varphi(x)\right| \le \int_0^{\infty} \left|K_{\alpha/2}^{(\ell)}(\eta)\right| \cdot \left|\left(G^{[m]}\varphi\right)(x,\varepsilon\eta) - \varphi(x)\right| d\eta ,$$

by means of Minkowskii inequality we get

$$\left\|\left(D_{1,\varepsilon}^{\alpha} J^{\alpha} \varphi\right) - \varphi\right\|_{L_p} \le \int_0^{\infty} \left|K_{\alpha/2}^{(\ell)}(\eta)\right| \cdot \left\|\left(G^{[m]}\varphi\right)(x,\varepsilon\eta) - \varphi(x)\right\|_{L_p} d\eta .$$

Estimate the value $i_0 = \left\|\left(G^{[m]}\varphi\right)(x,\varepsilon\eta) - \varphi(x)\right\|_{L_p}$ for fixed $\eta > 0$. We have (using the denotations of [6])

$$i_0 = \left\|e^{-\varepsilon\eta}(G\varphi)(x,\varepsilon\eta) - \varphi(x)\right\|_{L_p} \le \left\|e^{-\varepsilon\eta} \cdot (G\varphi)(x,\varepsilon\eta) - \right.$$

$$\left. - (G\varphi)(x,\varepsilon\eta)\right\|_{L_p} + \left\|(G\varphi)(x,\varepsilon\eta) - \varphi(x)\right\|_{L_p} = i_1 + i_2 .$$

Since $1 - e^{-\varepsilon\eta} \le e \cdot \varepsilon\eta$ and $\left\|(G\varphi)(x,\varepsilon\eta)\right\|_{L_p} \le \|\varphi\|_{L_p}$, then

$$i_1 \le e \cdot \|\varphi\|_{L_p} \cdot \varepsilon\eta = c_1 \cdot \varepsilon\eta ,$$

$$i_2 = \left\|(G\varphi)(x,\varepsilon\eta) - \varphi(x)\right\|_{L_p} = \left\|\int_{R^n} W(y,\varepsilon\eta)[\varphi(x-y) - \varphi(x)]dy\right\|_{L_p} \le$$

$$\le \int_{R^n} W(y,\varepsilon\eta)\left\|\varphi(x-y) - \varphi(x)\right\|_{L_p(R^n(x))}dy = \int_{|y|<\delta}\ldots + \int_{|y|>\delta}\ldots = i_3 + i_4 .$$

Denoting

$$g(x) = \begin{cases} \left\|\varphi(x-y) - \varphi(x)\right\|_{L_p(R^n(x))}, & \text{for } |y| \le \delta \\ 0 & , \text{for } |y| > \delta \end{cases}$$

we have

$$i_3 = \int_{R^n} W(y,\varepsilon\eta) \cdot g(y)dy \le c_2 \cdot \int_0^{\infty} \widetilde{W}(r,\varepsilon\eta) \cdot r^{n+\beta-1}dr,$$

where $\widetilde{W}(r,\varepsilon\eta) = W(y,\varepsilon\eta)|_{|y|=r}$.

In [6] we have shown that

$$\int_0^{\infty} \widetilde{W}(r,\varepsilon\eta) \cdot r^{n+\beta-1}dr = c_3 \cdot (\varepsilon\eta)^{\beta} .$$

So, $i_3 = c_4 \cdot (\varepsilon\eta)^{\beta}$, where c_4 is independent of ε and η. Estimate i_4. We have

$$i_4 \le \int_{|y|>\delta} W(y,\varepsilon\eta) \cdot \left\|\varphi(x-y) - \varphi(x)\right\|_{L_p(R^n(x))}dy \le 2\|\varphi\|_{L_p} \cdot \int_{|y|>\delta} W(y,\varepsilon\eta)dy,$$

and since

$$\int_{|y|>\delta} W(y,\varepsilon\eta)dy \le c_5 \cdot \varepsilon\eta .$$

then $i_4 \le c_6 \cdot \varepsilon\eta$, where c_6 is independent of ε and η. Now taking into account the asymptotics $K_{\alpha/2}^{(\ell)}(\eta)$ and the condition $\ell > \dfrac{\alpha}{2} + 1$, for small $\varepsilon > 0$ we have

$$\left\| D_\varepsilon^\alpha J^\alpha \varphi - \varphi \right\|_{L_p} \leq \int_0^\infty \left| K_{\alpha/2}^{(\ell)}(\eta) \right| \cdot \left(c_7 \cdot \varepsilon\eta + c_8 \cdot (\varepsilon\eta)^\beta \right) d\eta \leq$$

$$\leq \varepsilon^\beta \cdot \int_0^\infty \left| K_{\alpha/2}^{(\ell)}(\eta) \right| \cdot \left(c_7 \cdot \eta + c_8 \cdot \eta^\beta \right) d\eta = c_9 \cdot \varepsilon^\beta.$$

The theorem is proved.

In conclusion we note that the following theorem is valid

Theorem 4. *Let the density function* $\varphi \in L_p$ *and* $1 \leq p < \dfrac{n}{\alpha}$. *Then*

$$\left\| \widetilde{D}_\varepsilon^\alpha J^\alpha \varphi - \varphi \right\|_{L_p} = 0(\varepsilon^\beta), \quad \varepsilon \to 0.$$

In conclusion note that as it was shown in papers [6], [7] by B.S. Rubin, for $\varepsilon \to 0$ limit of construction D_ε^α and respectively $\widetilde{D}_\varepsilon^\alpha$, regarded in a definite sense transforms Bessel (respectively, Riesz) potentials. Here we are to note that in B.S.Rubin's indicated papers a natural condition $\ell > \dfrac{\alpha}{2}$ is imposed on ℓ. But we impose the condition

$\ell > \dfrac{\alpha}{2} + 1$.

I express my gratitude to corresponding member of AS A.D. Hajiyev for his help and discussion.

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