

**APPLIED PROBLEMS OF MATHEMATICS AND MECHANICS****AGAYEVA Ch.A., ALLAHVERDIYEVA J.J.****ON A STOCHASTIC OPTIMAL CONTROL WITH VARIABLE DELAY****Abstract**

The stochastic optimal control problem with variable delay is considered. The necessary condition of optimality is obtained when the coefficient of diffusion depends on control.

In [4] the necessary condition of optimality for stochastic control systems with constant delay was obtained. In [5], [6] the principle of maximum for stochastic control systems with variable delay was obtained under the condition that the coefficient of diffusion does not depend on control. In the present work the principle of maximum is proved for the stochastic control problem with variable delay when the diffusion coefficient depends on control.

Let  $(\Omega, \mathcal{F}, P)$  be complete probability space with the determined on it non-decreasing flow  $\sigma$ -algebras  $\{F^t, t \in [t_0, t_1] | F^t \subset \mathcal{F}\}$  and  $\overline{F^t} = \bar{\sigma}(W_s, t_0 \leq s \leq t)$ .  $L_F^2(t_0, t_1; R^n)$  is the space of measurable by  $(t, \omega)$  and adapted processes  $x: [t_0, t_1] \times \Omega \rightarrow R^n$  such that  $E \int_{t_0}^{t_1} |x_t|^2 dt < +\infty$ .  $L_F^2 C(t_0, t_1; R^n)$  is the space of functions  $x_t \in L_F^2(t_0, t_1; R^n)$  with a.c. (almost certainly) continuous trajectories.

Let us consider following stochastic system with variable delay

$$dx_t = g(x_t, x_{t-h(t)}, u_t, t)dt + \sigma(x_t, x_{t-h(t)}, u_t, t)dW_t, \quad t \in (t_0, t_1], \quad (1)$$

$$x_t = \Phi(t), \quad t \in [-h(t_0), t_0], \quad h(t) > 0, \quad (2)$$

$$u_t \in U \equiv \left\{ u \in L_F^2(t_0, t_1; R^n) | u_t \in U_g \subset R^m, \text{a.c.} \right\}. \quad (3)$$

Here  $U_g$  is non-empty bounded set.

Let on the set of admissible controls  $U$  it is required to minimize the functional

$$J(u) = E \left\{ p(x_{t_1}) + \int_{t_0}^{t_1} l(x_t, u_t, t)dt \right\}. \quad (4)$$

Here  $x(\cdot) \in L_F^2(t_0, t_1; R^n)$ ,  $\Phi(\cdot) \in L_F^2 C(-h(t_0), t_0; R^n)$ ,  $h(t)$  is a non-random function and  $\frac{dh(t)}{dt} \leq 1 - \varepsilon$ , for  $\forall \varepsilon > 0$ .

Suppose, that the following conditions are fulfilled:

- I. functions  $l(x, u, t)$ ,  $g(x, y, u, t)$ ,  $\sigma(x, y, u, t)$  are continuous by the population of arguments and:

$$(l, g, \sigma): R^n \times R^m \times R^m \times [t_0, t_1] \rightarrow R \times R^n \times R^{m \times m};$$

- II. for the fixed  $(t, u)$  function  $(l, g, \sigma)$  is differentiated by  $(x, y)$  and satisfies the condition:

$$(1+|x|+|y|)^{-1}(|g(x, y, u, t)| + |\sigma(x, y, u, t)| + |g_x(x, y, u, t)| + |\sigma_x(x, y, u, t)| + |g_y(x, y, u, t)| + |\sigma_y(x, y, u, t)|) \leq N,$$

$$(1+|x|)^{-1}(|l(x, u, t)| + |l_x(x, u, t)|) \leq N.$$

III. function  $p(x): R^n \rightarrow R$  is continuous differentiated and satisfies the condition:

$$|p(x)| + |p_x(x)| \leq N(1+|x|),$$

where  $N$  is some constant.

Let us reduce the definition, which will be used further.

**Definition [3].**  $L(x, X)$  is named a star neighborhood of point  $x$  with respect to set  $X$  if:

$$L(x, X) = \{y : y \in X, x + \varepsilon(y - x) \in X, \text{ for } \forall \varepsilon < \varepsilon_0, \varepsilon > 0\}. \quad (5)$$

The following result is obtained

**Theorem 1.** Let conditions I-III be fulfilled and  $(x_t^0, u_t^0)$  be the solution of problem (1)-(4). Then there exist random processes  $\psi_t \in L_p^2 C(t_0, t_1; R^n)$  and  $\beta_t \in L_p^2 C(t_0, t_1; R^{mn})$  which are the solution of following stochastic equation:

$$\begin{cases} d\psi_t = -[H_x(x_t^0, x_{t-h(t)}^0, u_t^0, \psi_t, t) + H_y(x_t^0, x_{t-h(t)}^0, u_t^0, \psi_t, z)]_{z=s(t)} dt + \beta_t dW_t, \\ \quad t_0 \leq t < t_1 - h(t), \\ d\psi_t = H_x(x_t^0, x_{t-h(t)}^0, u_t^0, \psi_t, t) dt + \beta_t dW_t, \quad t_1 - h(t) \leq t < t_1, \\ \psi_{t_1} = -p_x(x_{t_1}^0) \end{cases} \quad (6)$$

and

$$\max_{u \in \Lambda(u^0)} H(x_t^0, x_{t-h(t)}^0, u, \psi_t, t) = H(x_t^0, x_{t-h(t)}^0, u_t^0, \psi_t, t), \text{ a.c.} \quad (7)$$

Here  $\tau(t) = t - h(t)$ ;  $t = s(\tau)$  is the solution of equation  $\tau = \tau(t)$ ;

$$H(x_t^0, x_{t-h(t)}^0, u_t^0, \psi_t, t) = \psi_t^* g(x_t^0, x_{t-h(t)}^0, u_t^0, t) + \beta_t^* \cdot \sigma(x_t^0, x_{t-h(t)}^0, u_t^0, t) - l(x_t^0, u_t^0, t),$$

and set  $\Lambda(u^0)$  (see [3]) is determined by following way:

$$\begin{aligned} \Lambda(u^0) = \{u \in U_g : & (g(x_t^0, x_{t-h(t)}^0, u, t), \sigma(x_t^0, x_{t-h(t)}^0, u, t)) \in \\ & \in L(g(x_t^0, x_{t-h(t)}^0, u_t^0, t), \sigma(x_t^0, x_{t-h(t)}^0, u_t^0, t)), (g(x_t^0, x_{t-h(t)}^0, U_g, t), \\ & \sigma(x_t^0, x_{t-h(t)}^0, U_g, t))\}, \end{aligned} \quad (8)$$

where  $L(u^0, U)$  is star neighborhood of point  $u^0$  with respect to set  $U$ .

**Proof.** Let's  $\bar{u}_t = u_t^0 + \Delta u_t$ ,  $\bar{x}_t = x_t^0 + \Delta x_t$  are some admissible control and the corresponding trajectory. Then it is clear, that

$$\begin{cases} d\Delta x_t = d(\bar{x}_t - x_t^0) = \{\Delta_{\bar{u}} g(x_t^0, x_{t-h(t)}^0, u_t^0, t) + g_x(x_t^0, x_{t-h(t)}^0, u_t^0, t)\Delta x_t + \\ + g_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)\Delta x_{t-h(t)}\}dt + \{\sigma_x(x_t^0, x_{t-h(t)}^0, u_t^0, t)\Delta x_t + \sigma_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)\Delta x_{t-h(t)} + \\ + \Delta_{\bar{u}} \sigma(x_t^0, x_{t-h(t)}^0, u_t^0, t)\}dW_t + \eta_t^1, \quad t \in (t_0, t_1], \\ \Delta x_t = 0, \quad t \in [-h(t_0), t_0], \end{cases} \quad (9)$$

$$\eta_t^1 = \left\{ \int_0^1 [g_x(x_t^0 + \mu_1 \Delta x_t, \bar{x}_{t-h(t)}, \bar{u}_t, t) - g_x(x_t^0, x_{t-h(t)}^0, \bar{u}_t, t)] \Delta x_t d\mu_1 + \int_0^1 [g_y(x_t^0, x_{t-h(t)}^0, u_t^0, t) \Delta x_t d\mu_1 + \right. \\ \left. + \mu_2 \Delta x_{t-h(t)}, \bar{u}_t, t) - g_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)] \Delta x_{t-h(t)} d\mu_2 \right\} dt + \left\{ \int_0^1 [\sigma_x(x_t^0 + \mu_1 \Delta x_t, \bar{x}_{t-h(t)}, \bar{u}_t, t) - \right. \\ \left. - \sigma_x(x_t^0, x_{t-h(t)}^0, \bar{u}_t, t)] \Delta x_t d\mu_1 + \int_0^1 [\sigma_y(x_t^0, x_{t-h(t)}^0 + \mu_2 \Delta x_{t-h(t)}, \bar{u}_t, t) - \sigma_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)] \times \right. \\ \left. \times \Delta x_{t-h(t)}^0 d\mu_2 \right\} dW_t.$$

According to Ito's formula (see [1]) we have:

$$d(\psi_t^* \cdot \Delta x_t) = d\psi_t^* \cdot \Delta x_t + \psi_t^* d\Delta x_t + \{\beta_t^* \Delta_{\bar{u}} \cdot \sigma(x_t^0, x_{t-h(t)}^0, u_t^0, t) + \\ + \beta_t^* \sigma_x(x_t^0, x_{t-h(t)}^0, u_t^0, t) \times \Delta x_t + \beta_t^* \sigma_y(x_t^0, x_{t-h(t)}^0, u_t^0, t) \Delta x_{t-h(t)}\} dt. \quad (10)$$

Increment of functional (4) along the admissible control has the form:

$$\Delta_{\bar{u}} J(u^0) = J(\bar{u}) - J(u_0) = E \left\{ p(\bar{x}_{t_1}) - p(x_{t_1}^0) + \int_{t_0}^{t_1} [l(\bar{x}_t, \bar{u}_t, t) - l(x_t^0, u_t^0, t)] dt \right\}.$$

Taking into account (9) and (10) for increment of functional we obtain:

$$\Delta_{\bar{u}} J(u^0) = - \int_{t_0}^{t_1} [\psi_t^* \Delta_{\bar{u}} g(x_t^0, x_{t-h(t)}^0, u_t^0, t) + \beta_t^* \Delta_{\bar{u}} \sigma(x_t^0, x_{t-h(t)}^0, u_t^0, t) - \Delta_{\bar{u}} l(x_t^0, u_t^0, t)] dt - \\ - E \int_{t_0}^{t_1} [d\psi_t^* + \psi_t^* g_x(x_t^0, x_{t-h(t)}^0, u_t^0, t) + \beta_t^* \sigma_x(x_t^0, x_{t-h(t)}^0, t) - l_x(x_t^0, u_t^0, t)] \Delta x_t dt - \\ - E \int_{t_0}^{t_1} \Delta x_{t-h(t)} [\psi_t^* g_y(x_t^0, x_{t-h(t)}^0, u_t^0, t) + \beta_t^* \sigma_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)] dt + \eta_t, \quad (11)$$

where

$$\eta_t = E \int_0^1 [p_x^*(x_{t_1}^0 + \mu_1 \Delta x_{t_1}) - p_x^*(x_{t_1}^0)] \Delta x_{t_1} d\mu_1 + E \int_{t_0}^{t_1} [l_x^*(x_t^0 + \mu_1 \Delta x_t, \bar{u}_t, t) - l_x^*(x_t^0, u_t^0, t)] \times \\ \times \Delta x_t d\mu_1 \} dt + E \int_{t_0}^{t_1} \left\{ \int_0^1 \psi_t^* [g_x(x_t^0 + \mu_1 \Delta x_t, \bar{x}_{t-h(t)}, u_t^0, t) - g_x(x_t^0, \bar{x}_{t-h(t)}, u_t^0, t)] \Delta x_t d\mu_1 + \right. \\ \left. + \int_0^1 \psi_t^* [g_y(x_t^0, x_{t-h(t)}^0 + \mu_2 \Delta x_{t-h(t)}, u_t^0, t) - g_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)] \Delta x_{t-h(t)} d\mu_2 \right\} dt + \\ + E \int_{t_0}^{t_1} \left\{ \int_0^1 \beta_t^* [\sigma_x(x_t^0 + \mu_1 \Delta x_t, \bar{x}_{t-h(t)}, u_t^0, t) - \sigma_x(x_t^0, \bar{x}_{t-h(t)}, u_t^0, t)] \Delta x_t d\mu_1 + \int_0^1 \beta_t^* \times \right. \\ \left. \times [\sigma_y(x_t^0, x_{t-h(t)}^0 + \mu_2 \Delta x_{t-h(t)}, u_t^0, t) - \sigma_y(x_t^0, x_{t-h(t)}^0, u_t^0, t)] \Delta x_{t-h(t)} d\mu_2 \right\} dt.$$

Let random processes  $\psi_t \in L_F^2 C(t_0, t_1; R^n)$  and  $\beta_t \in L_F^2 C(t_0, t_1; R^{n \times n})$  are solutions of adjoint equation (6). Assume that (7) is not fulfilled, i.e. for some  $\theta, u \in \Lambda(u_\theta^0)$  the inequality is fulfilled:

$$H(x_\theta^0, x_{\theta-h(\theta)}^0, u, \psi_\theta, \theta) - H(x_\theta^0, x_{\theta-h(\theta)}^0, u_\theta^0, \psi_\theta, \theta) = \alpha < 0. \quad (12)$$

According to the definition of set  $\Lambda(u_\theta^0)$  there exist the sequence of numbers  $\{\varepsilon_i\}$ ,  $\varepsilon_i \rightarrow 0$ ,  $\varepsilon_i > 0$  and the sequence of vectors  $\{u_i\}$ ,  $u_i \in U$  such that

$$\Delta_{u_i} g(x_\theta^0, x_{\theta-h(\theta)}^0, u_\theta^0, \theta) = \varepsilon_i \Delta_u g(x_\theta^0, x_{\theta-h(\theta)}^0, u_\theta^0, \theta), \quad (13)$$

$$\Delta_{u_i} \sigma(x_\theta^0, x_{\theta-h(\theta)}^0, u_\theta^0, \theta) = \varepsilon_i \Delta_u \sigma(x_\theta^0, x_{\theta-h(\theta)}^0, u_\theta^0, \theta). \quad (14)$$

Let us consider the variations of the form:

$$\Delta_{u_i, \alpha_i, \theta} u_i^0 = \begin{cases} u_i - u_i^0, & t \in [\theta, \theta + \alpha_i], u_i \in L_F^2(\theta, \theta + \alpha_i; R^m) \\ 0, & t \notin [\theta, \theta + \alpha_i], \alpha_i < h(\theta) \end{cases} \quad (15)$$

We denote the trajectories corresponding to the variations (15) by  $x'_t = x_t^0 + \Delta_i x_t^0$ . Let us estimate the quantity  $E|\Delta_i x_t^0|^2$ .

Using Gronuoll's inequality, by (13) and (14) it is not difficult to show that:

$$E|\Delta_i x_t^0|^2 = 0, \quad \forall t \in [0, \theta],$$

$$E|\Delta_i x_t^0|^2 \leq N_1 \varepsilon_i^2 \alpha_i, \quad \forall t \in [\theta, \theta + \alpha_i],$$

$$E|\Delta_i x_t^0|^2 \leq N_2 \varepsilon_i^2 \alpha_i, \quad \forall t \in [\theta + \alpha_i, \theta + h(\theta)].$$

Therefore:  $E|\Delta_i x_t^0|^2 \leq N \varepsilon_i^2 \alpha_i, \quad \forall t \in [\theta, \theta + h(\theta)]$ .

Here:  $N_1, N_2, N_3$  are the constants obtained at the result of the transformations.

Let us consider consequentially the segments  $[\theta + (j-1)h(\theta), \theta + jh(\theta)]$ ,  $j = \overline{1, m}$ ,  $\theta + mh(\theta) \geq t_1$ , dividing each of them into the segments  $[\theta + (j-1)h(\theta), \theta + (j-1)h(\theta) + \alpha_i]$  and  $[\theta + (j-1)h(\theta) + \alpha_i, \theta + jh(\theta)]$  we can prove validity of following estimations:

$$E|\Delta_i x_t^0|^2 \leq N \varepsilon_i^2 \alpha_i, \quad t \in [\theta + (j-1)h(\theta), \theta + jh(\theta)], \quad j = \overline{1, m}.$$

As far as random processes  $\psi_i, \beta_i$  are the solutions of (6) for increment of the functional we obtain following:

$$\begin{aligned} \Delta_{u_i} J(u^0) &= -E \int_0^{\theta+\alpha_i} [\psi_i^* \Delta_{u_i} g(x_i^0, x_{i-h(t)}^0, u_i^0, t) + \beta_i^* \Delta_{u_i} \sigma(x_i^0, x_{i-h(t)}^0, u_i^0, t)] dt + \eta_\theta = \\ &= -\alpha_i \varepsilon_i E[\psi_\theta^* \Delta_u g(x_\theta^0, x_{\theta-h(\theta)}^0, u_\theta^0, \theta) + \beta_\theta^* \Delta_u \sigma(x_\theta^0, x_{\theta-h(\theta)}^0, u_\theta^0, \theta)] + o(\varepsilon_i \alpha_i). \end{aligned}$$

Now by choose of numbers  $\alpha_i$  and using assumptions (12) for sufficient large  $i$  we obtain

$$\Delta_{u_i} J(u^0) < 0.$$

And that contradicts to optimality of control  $u_i^0$ . Theorem 1 has been proved.

From the aboveproved theorem the series of other results is obtained. For example, it is valid

**Theorem 2.** *Let the conditions of Theorem 1 be fulfilled and moreover, assume that the functions  $g(x, y, u, t)$  and  $\sigma(x, y, u, t)$  are convex with respect to  $u$ . Then there exist random processes  $\psi_i \in L_F^2 C(t_0, t_1; R^n)$  and  $\beta_i \in L_F^2 C(t_0, t_1; R^{mn})$  which are the solutions of equation (6) and*

$$\max_{u \in U_g} H(x_t^0, x_{t-h(t)}^0, u, \psi_t, t) = H(x_t^0, x_{t-h(t)}^0, u_t^0, \psi_t, t), \quad a.c.$$

**References**

- [1]. Гихман И.И., Скороход А.В. *Введение в теорию случайных процессов*. М., «Наука», 1977, 567с.
- [2]. Черноусько Ф.А., Калмановский В.Б. *Оптимальное управление при случайных возмущениях*. М., «Наука», 1978, 352с.
- [3]. Габасов Р., Кириллова Ф.М. *Принципы максимума в теории оптимального управления*. Минск, 1974, 272с.
- [4]. Агаева Ч.А. *Принцип максимума для выпуклой стохастической задачи оптимального управления с запаздыванием*. Известия АН Азербайджана, сер.физ.-мат. наук, 1994, №1-2, с. 42-46.
- [5]. Агаева Ч.А., Аллахвердиева Дж.Дж. *Необходимое условие оптимальности для стохастической системы управления с переменным запаздыванием*. Тезисы международной конференции «Динамические системы, устойчивость, управление, оптимизация». Минск, 1998, т. I, стр. 11-12.
- [6]. Агаева Ч.А., Аллахвердиева Дж.Дж. *Принцип максимума для одной задачи стохастического оптимального управления с переменным запаздыванием*. Материалы научной конференции «Вопросы функционального анализа и математической физики», посвященной 80-летию БГУ им. М.Э. Расул-заде, Баку, 1999, стр. 95-103.

**Agayeva Ch.A., Allahverdiyeva J.J.**

Baku State University named after E.M. Rasulzadeh.  
23, Z.I. Khalilov str., 370148, Baku, Azerbaijan.

Received February 8, 2000; Revised May 31, 2000.

Translated by Soltanova S.M.