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ON A PROBLEM OF CONTROL FOR WAVE EQUATION

Abstract

In the work the solution of a problem of optimal control for the wave equation is reduced to the solution of Fredholm integral equation of the second type.

Let the controlled process be described by the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f(x)\cdot \vartheta(t) \quad \text{in} \quad Q = (0,T)\times\Omega \tag{1}$$

with the initial and boundary conditions

$$u(0,x) = \varphi_0(x), \quad \frac{\partial u(0,x)}{\partial t} = \varphi_1(x), \quad u|_S = 0, \tag{2}$$

where $\Omega \subset R^n$ is the bounded domain with the smooth boundary Γ, $0 < T < \infty$, S is the lateral surface Q of the cylinder, Δ is Laplace operator.

It is requested to minimize the functional

$$J(\vartheta) = \frac{\gamma}{2}\int_0^T \vartheta^2(t)dt + \frac{1}{2}\int_0^T\int_\Omega \left[\left(\frac{\partial u(t,x)}{\partial t}\right)^2 + \sum_{i=1}^m \left(\frac{\partial u(t,x)}{\partial x_i}\right)^2\right] dxdt \tag{3}$$

in the class of controls $U_{ad} = L_2(0,T)$ for the restrictions (1)-(2), where $f(x) \in L_2(\Omega)$, $\varphi_0 \in \overset{\circ}{W}{}_2^1(\Omega)$, $\varphi_1 \in L_2(\Omega)$ are the given functions, $\gamma > 0$ is the given number. Let's note that problem (1)-(2) has the unique generalized solution in class $W_{2,0}^1(Q)$. It should remark that the minimization problem has the unique solution in $L_2(0,T)$. Really, problem (1)-(2) is linear, so the functional by ϑ is quadratic. Then reasoning as in [1] it is easy to show that (1)-(3) has the unique solution. It is clear, that the identity for function $u(t,x)$ is valid which has the quadratic summed in Q derivatives of the second order:

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^m\left(\frac{\partial u}{\partial x_i}\right)^2\right] - \sum_{i=1}^m \frac{\partial}{\partial x_i}\left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_i}\right] = \frac{\partial u}{\partial t}\left[\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^m \frac{\partial^2 u}{\partial x_i^2}\right]. \tag{4}$$

Let us note that smoothing the functions $f(x)$, $\vartheta(t)$, $\varphi_0(x)$, $\varphi_1(x)$ it is possible to achieve that the solution of problem (1)-(2) has the second quadratic summed derivatives in Q, this solution satisfies equation (1) almost everywhere in Q and for this solution identity (4) is valid.

Then integrating in domain $(0,t)\times\Omega$ identity (4) obtained for such solution and using equation (1) and then passing to the limit by smoothing parameter we have:

$$E(t,\vartheta) \equiv \frac{1}{2}\int_\Omega\left[\left(\frac{\partial u(t,x)}{\partial t}\right)^2 + \sum_{i=1}^m\left(\frac{\partial u(t,x)}{\partial x_i}\right)^2\right]dx = \frac{1}{2}\int_\Omega\left[\left(\frac{\partial u(0,x)}{\partial t}\right)^2 + \sum_{i=1}^m\left(\frac{\partial u(0,x)}{\partial x_i}\right)^2\right]dx + \int_0^t\int_\Omega \frac{\partial u(\tau,x)}{\partial \tau} f(x)\vartheta(\tau)dxd\tau, \tag{5}$$

where $E(t,\vartheta)$ is vibration energy of the system at the time moment t. Here we considered

$$\left.\frac{\partial u}{\partial t}\right|_S = 0 .$$

It is clear that the solution of problem (1)-(2) can be represented in the form:

$$u(t,x) = u^*(t,x) + u_0(t,x), \tag{6}$$

where

$$u^*(t,x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} \int_0^t \int_\Omega \vartheta(s) f(\xi) W_k(\xi) \sin\sqrt{\lambda_k}(t-s) d\xi ds W_k(x)$$

is the solution of problem (1)-(2) with zero initial functions, and

$$u_0(t,x) = \sum_{k=1}^{\infty} (a_k \cos\lambda_k t + b_k \sin\lambda_k t) W_k(x)$$

is the solution of problem (1)-(2) with zero right-hand side, λ_k , $W_k(x)$ are eigenvalues and eigenfunctions of the spectral problem:

$$-\Delta W = \lambda W \ , \qquad W|_\Gamma = 0 , \tag{7}$$

$$a_k = \int_\Omega \varphi_0(x) W_k(x) dx \ , \quad b_k = \frac{1}{\sqrt{\lambda_k}} \int_\Omega \varphi_1(x) W_k(x) dx ,$$

moreover we can consider that the system $\{W_k(x)\}$ is ortonormalized in $L_2(\Omega)$.

Substituting (6) in (5) we obtain

$$E(t,\vartheta) = \frac{1}{2} \int_\Omega \left[\left| \sum_{i=1}^{m} \frac{\partial \varphi_0}{\partial x_i} \right|^2 + \varphi_1^2 \right] dx + \int_0^t \int_\Omega \frac{\partial u^*}{\partial \tau} f(x) \vartheta(\tau) dx d\tau +$$

$$+ \int_0^t \int_\Omega \frac{\partial u_0}{\partial \tau} f(x) \vartheta(\tau) dx d\tau .$$

Let us denote

$$F(\tau) = \int_\Omega f(x) \frac{\partial u_0(\tau,x)}{\partial \tau} dx .$$

Taking into account that

$$\frac{\partial u^*(t,x)}{\partial t} = \sum_{k=1}^{\infty} \int_0^t \int_\Omega f(\xi) \vartheta(\tau) W_k(\xi) \cos\sqrt{\lambda_k}(t-\tau) d\xi d\tau W_k(x),$$

we have

$$\int_\Omega f(x) \frac{\partial u^*(\tau,x)}{\partial \tau} dx = \sum_{k=1}^{\infty} f_k^2 \int_0^\tau \vartheta(s) \cos\sqrt{\lambda_k}(\tau - s) ds = \int_0^\tau K(\tau,s) \vartheta(s) ds ,$$

where the denotations are accepted:

$$f_k = \int_\Omega f(x) W_k(x) dx \ , \ \ K(\tau,s) = \sum_{k=1}^{\infty} f_k^2 \cos\sqrt{\lambda_k}(\tau - s) .$$

By Parseval's equality it is clear, that

$$|K(\tau,s)| \le \sum_{k=1}^{\infty} f_k^2 = \|f\|^2_{L_2(\Omega)} < \infty$$

and hence it follows that $K(\tau,s)$ is a continuous function in domain $[0,T]\times[0,T]$.

Therefore, we reduce the expression of functional (3) to the form

$$J(\vartheta)=\frac{\gamma}{2}\int_0^T \vartheta^2(t)dt+\int_0^T E(t,\vartheta)dt=c+\frac{\gamma}{2}\int_0^T \vartheta^2(t)dt+\int_0^T\left(\int_0^t\int_\Omega \frac{\partial u(\tau,x)}{\partial \tau}f(x)\vartheta(\tau)dxd\tau\right)dt=$$

$$=c+\frac{\gamma}{2}\int_0^T \vartheta^2(t)dt+\int_0^T\int_0^t F(\tau)\vartheta(\tau)d\tau dt+\int_0^T\int_0^t\left(\int_0^\tau K(\tau,s)\vartheta(s)ds\right)\vartheta(\tau)d\tau dt, \tag{8}$$

where c is constant.

If we change the order of integration in the last two components in (8), we have

$$J(\vartheta)=c+\int_0^T\left\{\frac{\gamma}{2}\vartheta^2(t)+(T-t)\vartheta(t)\left[F(t)+\int_0^t K(t,\tau)\vartheta(\tau)d\tau\right]\right\}dt.$$

If some control $\vartheta(t)\in U_{ad}$ is the solution of the minimization problem, then for it the condition is fulfilled:

$$\delta J(\vartheta;h)=0 \qquad \forall h\in L_2(0,T),$$

where $\delta J(\vartheta;h)$ is the first variation of functional $J(\vartheta)$.

Then calculating the first variation of the functional we have:

$$\delta J(\vartheta;h)=\int_0^T\left\{\gamma\,\vartheta(t)h(t)+(T-t)F(t)h(t)+(T-t)\int_0^t K(t,\tau)\vartheta(t)d\tau\cdot h(t)+\right.$$

$$\left.+(T-t)\vartheta(t)\int_0^t K(t,\tau)h(\tau)d\tau\right\}dt=0,\ \ \forall h\in L_2(0,T),$$

or

$$\int_0^T\left\{\gamma\,\vartheta(t)+(T-t)F(t)+\int_0^t \vartheta(\tau)(T-t)K(t,\tau)d\tau+\int_t^T \vartheta(\tau)(T-\tau)K(\tau,t)d\tau\right\}h(t)dt=0,$$

$$\forall h\in L_2(0,T).$$

Hence by virtue of arbitrarity of $h(t)$ we have

$$\gamma\vartheta(t)+\int_0^t \vartheta(\tau)(T-t)K(t,\tau)d\tau+\int_t^T \vartheta(\tau)(T-\tau)K(\tau,t)d\tau=-(T-t)F(t) \tag{9}$$

for almost all $t\in[0,T]$.

If to take one more denotation

$$\phi(t,\tau)=\begin{cases}(T-t)K(t,\tau), & 0\le\tau\le t,\\ (T-\tau)K(\tau,t), & t\le\tau\le T,\end{cases}$$

then for finding the optimal control equation (9) is rewritten in the form

$$\vartheta(t)+\frac{1}{\gamma}\int_0^T \phi(t,\tau)\vartheta(\tau)d\tau=\frac{1}{\gamma}(t-T)F(t). \tag{10}$$

Therefore, for optimal control finding we obtain Fredholm integral equation of second type.

Theorem. *Let the above given conditions be fulfilled for data of problem (1)-(3). If $\vartheta_0(t)\in L_2(0,T)$ is the optimal control in this problem, then it satisfies the integral equation (10).*

If $\varphi_0(x)\equiv\varphi_1(x)\equiv 0$, then $u_0(t,x)\equiv 0$, hence it follows that $F(t)\equiv 0$. Then equation (10) has zero solution $\vartheta_0(t)\equiv 0$. As far as in this case problem (1)-(2) has unique zero solution the optimal control $\vartheta_0(t)\equiv 0$ gives to the functional $J(\vartheta)$ zero value.

Remark. Analogous problem in the one-dimensional case and when $f(x) \equiv x$ was solved in [2].

References

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