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ON A CAUCHY PROBLEM WITH INVERSE CURENT OF TIME FOR THE SYSTEM OF EQUATIONS OF MOTION OF VISCO-ELASTIC MEDIUM

Abstract

In present work in contrast to the method of quasi-transformation to the solution of Cauchy problem with inverse current of time for weak parabolic systems more rational method is used which represents combination of M.L.Rasulov's method of contour integral [2] and A.N. Tikhonov's method of regularisation [3,4] For the first time this method for similar problems was used in [5].

Let us consider the system

$$\frac{\partial^2 u}{\partial t^2}=\left(\mu+\mu'\frac{\partial}{\partial t}\right)\Delta u+\left[v+\mu+(v'+\mu')\frac{\partial}{\partial t}\right]\partial\partial' u \qquad (1)$$

known in reference as the equation of motion of visco-elastic medium, where v,μ are Lyame's constants and v',μ' are viscous or dissipative constants, $x=(x_1,x_2,x_3)$, $\partial'=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\right)$, and ∂ is the corresponding column of differential operators $\frac{\partial}{\partial x_k}$ $(k=1,2,3)$, Δ is Laplace threemensional operator, $u(x,t)$ is the threemensional vector-function.

Let the member $T>0$ and some continuous in space E_3 vector-functions $\varphi_0(x)$ and $\varphi_1(x)$ be given. We will name by Cauchy problem with inverse current of time for system (1) the finding of the family of vector-functions $\{u_\alpha(x,t)\}$ (α is positive parameter), satisfying for $t\in[0,T]$, $x\in E_3$ system (1) and the conditions:

$$\lim_{\alpha\to 0}u_\alpha(x,T)=\varphi_0(x),\quad \lim_{\alpha\to 0}\left.\frac{\partial u_\alpha(x,t)}{\partial t}\right|_{t=T}=\varphi_1(x) \qquad (2)$$

uniformly on every compact of space E_3.

We will name the problem of finding of the solution of the system with complex parameter

$$\Delta v(x,\lambda)+\frac{v+\mu+\lambda^2(v'+\mu')}{\mu+\lambda^2\mu'}\partial\partial' v(x,\lambda)-\frac{\lambda^4}{\mu+\lambda^2\mu'}v(x,\lambda)=F(x,\lambda), \qquad (3)$$

where

$$F(x,\lambda)+\frac{\lambda^2\varphi_0(x)+\varphi_1(x)}{\mu+\lambda^2\mu'} \qquad (4)$$

by the auxiliary problem corresponding to Cauchy problem with inverse current of time (1), (2).

For convenience we introduce the denotations

$$M_1(\lambda)=\frac{\lambda^2}{\sqrt{\mu+\lambda^2\mu'}},\ M_2(\lambda)=\frac{\lambda^2}{\sqrt{v+2\mu+\lambda^2(v'+2\mu')}} \qquad (5)$$

moreover under the radical signs in (5) we will understand that branch which for real λ receives positive values. In these denotations system (3) takes the form:

$$\Delta v(x,\lambda)+\left(\frac{M_1^2(\lambda)}{M_2^2(\lambda)}-1\right)\partial\partial' v(x,\lambda)-M_1^2(\lambda)v(x,\lambda)=F(x,\lambda). \qquad (6)$$

The solution of equation (6) for sufficient smooth vector-function $F(x,\lambda)$ has the form:

$$v(x,\lambda)=-\int_{E_3}P(x-\xi,\lambda)F(\xi,\lambda)dD_\xi, \qquad (7)$$

where $P(x,\lambda)$ is the fundamental matrix of the corresponding homogeneous system (6) for which we have the formula:

$$P(x,\lambda)=\frac{1}{4\pi|x|}\left\{\left[\left(1-\frac{1}{M_1(\lambda)|x|}-\frac{1}{M_1^2(\lambda)|x|^2}\right)e^{-M_1(\lambda)|x|}+\right.\right.$$
$$\left.+\left(\frac{M_2(\lambda)}{M_1^2(\lambda)|x|}+\frac{1}{M_1^2(\lambda)|x|^2}\right)e^{-M_2(\lambda)|x|}\right]\cdot E+$$
$$+\left[\left(-1+\frac{3}{M_1(\lambda)|x|}+\frac{3}{M_1^2(\lambda)|x|^2}\right)e^{-M_1(\lambda)|x|}+\right.$$
$$\left.\left.+\left(\frac{M_2^2(\lambda)}{M_1^2(\lambda)}-\frac{3M_2(\lambda)}{M_1^2(\lambda)|x|}-\frac{3}{M_1^2(\lambda)|x|^2}\right)e^{-M_2(\lambda)|x|}\right]\frac{xx'}{|x|^2}\right\}, \qquad (8)$$

where $|x|$ is the length of vector-column x, E is the unitary matrix of the third order. Let us denote by R_δ the domain of values λ satisfying the inequalities:

$$|\arg\lambda|\le\frac{3\pi}{8}+\delta,\ |\lambda|\ge R, \qquad (9)$$

where $\delta>0$ is sufficient small and $R>0$ is a sufficient big number.

Let S be the non-bounded open contour arranged in domain R_δ whose sufficient far part coincides with continuous of the rays

$$\arg\lambda=\pm\left(\frac{3\pi}{8}+\delta\right).$$

Theorem. *If vector-functions $\varphi_k(x)$ are continuous and bounded with their derivatives up to the order $3-2k$, then there exists the family of vector-functions $\{u_\alpha(x,t)\}$, $\alpha>0$ represented in the form*

$$u_\alpha(x,t)=-\frac{1}{\pi i}\int_S e^{\lambda^2(t-T)-\alpha\lambda^4}\lambda d\lambda\int_{E_3}P(x-\xi,\lambda)F(\xi,\lambda)dD_\xi, \qquad (10)$$

which for $t\in[0,T]$, $x\in E_3$ satisfy system (1) and for $t=T$ conditions (2) uniformly on every compact of threemensional space E_3.

Proof. As it is seen from (8) and by choose of contour S integral (10) admits differentiation twice by x and once by t for $\alpha>0$.

Consequently we have:

$$\left[\frac{\partial^2}{\partial t^2}-\left(\mu+\mu'\frac{\partial}{\partial t}\right)\Delta-\left(\nu+\mu+(\nu'+\mu')\frac{\partial}{\partial t}\right)\partial\partial' u_\alpha(x,t)\right]=$$

$$=\frac{1}{\pi i}\int_S e^{\lambda^2(t-T)-\alpha\lambda^4}\lambda d\lambda\left(\mu+\mu'\lambda^2\right)\left[\Delta+\left(\frac{M_1^2(\lambda)}{M_2^2(\lambda)}-1\right)\partial\partial'-M_1^1(\lambda)\right]\times$$

$$\times\int_{E_3}P(x-\xi,\lambda)F(\xi,\lambda)dD_\xi=$$

$$=\frac{-1}{\pi i}\int_S e^{\lambda^2(t-T)-\alpha\lambda^4}\left[\lambda^2\varphi_0(x)+\varphi_1(x)\right]\lambda d\lambda\equiv 0\,.$$

In order to finish proving we must check the condition (2). According to (7) and (8) let us represent $\mathrm{v}(x,\lambda)$ in the form of sum:

$$\mathrm{v}(x,\lambda)=-\int_{E_3}\frac{E}{4\pi|x-\xi|}e^{-M_1(\lambda)|x-\xi|}\frac{\lambda^2\varphi_0(\xi)}{\mu+\lambda^2\mu'}dD_\xi-$$

$$-\int_{E_3}P_1(x-\xi,\lambda)\frac{\lambda^2\varphi_0(\xi)}{\mu+\lambda^2\mu'}dD_\xi-\int_{E_3}P(x-\xi,\lambda)\frac{\varphi_1(\xi)}{\mu+\lambda^2\mu'}dD_\xi=$$

$$=-\left(\mathrm{v}_1(x,\lambda)+\mathrm{v}_2(x,\lambda)+\mathrm{v}_3(x,\lambda)\right)\ ,\tag{11}$$

where $P_1(x-\xi,\lambda)=P(x-\xi,\lambda)-\dfrac{E}{4\pi|x-\xi|}e^{-M_1(\lambda)|x-\xi|}$.

By virtue of the conditions of the theorem $\dfrac{\lambda^2\varphi_0(\xi)}{\mu+\mu'\lambda^2}$ is bounded for all $\lambda\in R_\delta$.

Then from (8) it is easy to get that the analytic by λ function $\mathrm{v}_2(x,\lambda)$ in domain R_δ satisfies the estimation:

$$\left|\mathrm{v}_2(x,\lambda)\right|\le\frac{C}{|\lambda|^3},\ \lambda\in R_\delta\,.\tag{12}$$

Further from (8) it is easy to receive that the analytic by λ function $\mathrm{v}_2(x,\lambda)$ in domain R_δ satisfies the estimation

$$\left|\mathrm{v}_3(x,\lambda)\right|\le\frac{C}{|\lambda|^4},\ \lambda\in R_\delta\,.\tag{13}$$

By virtue of (11) we get:

$$\lim_{\alpha\to0}u_\alpha(x,T)=\lim_{\alpha\to0}\left\{\frac{-1}{\pi i}\int_S e^{-\alpha\lambda^4}\lambda\mathrm{v}(\mathrm{x},\lambda)d\lambda\right\}=$$

$$=\frac{1}{\pi i}\lim_{\alpha\to0}\left\{\int_S e^{-\alpha\lambda^4}\lambda\mathrm{v}_1(\mathrm{x},\lambda)d\lambda+\int_S e^{-\alpha\lambda^4}\lambda\left(\mathrm{v}_2(\mathrm{x},\lambda)+\mathrm{v}_3(\mathrm{x},\lambda)\right)d\lambda\right\}.\tag{14}$$

Estimations (12) and (13) allow to pass to limit $\alpha\to0$ immediately under the sign of the second integral in the right-hand side of (14)

$$\lim_{\alpha\to0}u_\alpha(x,T)=\frac{1}{\pi i}\lim_{\alpha\to0}\int_S e^{-\alpha\lambda^4}\lambda\mathrm{v}_1(\mathrm{x},\lambda)d\lambda+\frac{1}{\pi i}\int_S\lambda\left(\mathrm{v}_2(\mathrm{x},\lambda)+\mathrm{v}_3(\mathrm{x},\lambda)\right)d\lambda\,.\tag{15}$$

It is obvious that

$$\frac{1}{\pi i}\int_S \lambda(\mathrm{v}_2(\mathrm{x},\lambda)+\mathrm{v}_3(\mathrm{x},\lambda))d\lambda=\frac{1}{\pi i}\lim_{v\to\infty}\int_S \lambda(\mathrm{v}_2(\mathrm{x},\lambda)+\mathrm{v}_3(\mathrm{x},\lambda))d\lambda=$$
$$=\lim_{v\to\infty}\frac{1}{\pi i}\int_{O_{v_1}}\lambda(\mathrm{v}_2(\mathrm{x},\lambda)+\mathrm{v}_3(\mathrm{x},\lambda))d\lambda=0, \tag{16}$$

where by O_{v_1} the part of the arc of the circumference is denoted whose radius r_v ($r_v\to\infty$ for $v\to\infty$) with the centre in the origin of coordinates of λ-plane arranged in domain R_δ and by S_v the part of contour S arranged in O_v.

Taking into account (16) from (15) we obtain:

$$\lim_{\alpha\to 0}u_\alpha(x,T)=\frac{1}{\pi i}\lim_{\alpha\to 0}\int_S e^{-\alpha\lambda^4}\lambda\mathrm{v}_1(\mathrm{x},\lambda)d\lambda=$$
$$=\frac{1}{\pi i}\lim_{\alpha\to 0}\int_S e^{-\lambda^4\alpha}\lambda d\lambda\int_{E_3}\frac{1}{4\pi|x-\xi|}e^{-M_1(\lambda)|x-\xi|}\frac{\lambda^2\varphi(\xi)}{\mu+\mu'\lambda^2}dD_\xi=$$
$$=\lim_{\alpha\to 0}\left\{\frac{1}{\pi i}\int_S\frac{\lambda}{\mu'}r^{-\alpha\lambda^4}d\lambda\int_{E_3}\frac{1}{4\pi|x-\xi|}e^{-\frac{\lambda}{\sqrt{\mu}}|x-\xi|}\varphi_0(\xi)dD_\xi+\right.$$
$$+\frac{1}{\pi i}\int_S\frac{\lambda}{\mu'}r^{-\alpha\lambda^4}d\lambda\int_{E_3}\frac{1}{4\pi|x-\xi|}\left[e^{-M_1(\lambda)|x-\xi|}-e^{-\frac{\lambda}{\sqrt{\mu}}|x-\xi|}\right]\varphi_0(\xi)dD_\xi-$$
$$\left.-\frac{1}{\pi i}\int_S\frac{\mu}{\mu'}\frac{\lambda e^{-\alpha\lambda^4}}{\mu+\lambda^2\mu'}d\lambda\int_{E_3}\frac{1}{4\pi|x-\xi|}e^{-M_1(\lambda)|x-\xi|}\varphi_0(\xi)dD_\xi\right\}. \tag{17}$$

In domain R_δ the representation is valid:

$$M_1(\lambda)=\frac{\lambda^2}{\sqrt{\mu+\mu'\lambda^2}}=\frac{\lambda}{\sqrt{\mu}}+o(\lambda^{-1}). \tag{18}$$

Taking into account (18) we obtain

$$\frac{1}{|x-\xi|}\left|e^{-M_1(\lambda)|x-\xi|}-e^{-\frac{\lambda}{\sqrt{\mu}}|x-\xi|}\right|\le Ce^{-\varepsilon|\lambda||x-\xi|}, \tag{19}$$

for all $\lambda\in R_\delta$ positive constants C and ε do not depend on λ. Using analyticity of the integrand function in the second integral in (17) and estimation (19) for $\alpha\to 0$ this integral tends to zero.

Consequently,

$$\lim_{\alpha\to 0}u_\alpha(x,T)=\lim_{\alpha\to 0}\frac{1}{\pi i}\int_S\frac{\lambda}{\mu'}e^{-\alpha\lambda^4}d\lambda\int_{E_3}\frac{e^{-\frac{\lambda}{\sqrt{\mu'}}|x-\xi|}}{4\pi|x-\xi|}\varphi_0(\xi)dD_\xi\,. \tag{20}$$

For all $\frac{\lambda}{\sqrt{\mu'}}\in R_\delta$ the equality is valid

$$\frac{1}{4\pi|x-\xi|}e^{-\frac{\lambda}{\sqrt{\mu'}}|x-\xi|}=\frac{1}{(2\pi)^3}\int_{E_3}\frac{e^{i(x-\xi,\sigma)}}{\frac{\lambda^2}{\mu'}+|\sigma|^2}dD_\sigma\,. \tag{21}$$

Substituting (21) in (20) and inverting the order of integration by λ and σ we obtain

$$\lim_{\alpha\to 0} u_\alpha(x,T) = \lim_{\alpha\to 0} \int_{E_3} \frac{1}{(2\pi)^3} \varphi_0(\xi) dD_\xi \int_{E_3} \left\{ \frac{1}{\pi i} \int_S \frac{\lambda e^{-\alpha\lambda^4} d\lambda}{\lambda^2 + \mu'|\sigma|^2} \right\} e^{i(x-\xi,\sigma)} dD_\sigma \tag{22}$$

Contour integral in (22) is calculated easy:

$$\frac{1}{\pi i} \int_S \frac{\lambda e^{-\alpha\lambda^4}}{\lambda^2 + \mu'|\sigma|^2} d\lambda = e^{-\alpha\mu'^2} |\sigma|^4 .$$

Then

$$\lim_{\alpha\to 0} u_\alpha(x,T) = \lim_{\alpha\to 0} \int_{E_3} \varphi_0(\xi) dD_\xi \left\{ \frac{1}{(2\pi)^3} \int_{E_3} e^{i(x-\xi,\sigma) - \alpha\mu'^2|\sigma|^4} dD_\sigma = \right. \\ = \lim_{\alpha\to 0} \int_{E_3} \varphi_0(\xi) \mathcal{E}_\alpha(x-\xi) dD_\xi , \tag{23}$$

where

$$\mathcal{E}_\alpha(x-\xi) = \frac{1}{(2\pi)^3} \int_{E_3} e^{i(x-\xi,\sigma) - \alpha\mu'^2|\sigma|^4} dD_\sigma .$$

As in [5] it is possible to prove that

$$\frac{1}{(2\pi)^3} \int_{E_3} \mathcal{E}_\alpha(x-\xi) dD_\sigma = 1 . \tag{24}$$

Using formula (24) we can write

$$\int_{E_3} \varphi_0(\xi) \mathcal{E}_\alpha(x-\xi) dD_\xi - \varphi_0(x) = \\ = \int_{E_3} [\varphi_0(x-\xi) - \varphi_0(\xi)] \mathcal{E}_\alpha(\xi) dD_\xi .$$

Making change $\alpha^{1/4} = \beta,\ \xi = \beta \cdot t$, we obtain:

$$\int_{E_3} \mathcal{E}_\alpha(x-\xi) \varphi_0(\xi) dD_\xi - \varphi_0(x) = \\ = \int_{E_3} [\varphi_0(x-\beta t) - \varphi_0(x)] \mathcal{E}(t) dD_t , \tag{25}$$

where

$$\mathcal{E}(t) = \frac{1}{(2\pi)^3} \int_{E_3} e^{-\mu'|\sigma|^4 + i(t,\sigma)} dD_\sigma . \tag{26}$$

Giving $\eta > 0$ let us estimate the left-hand side f (25). It is obvious that there exists such $\rho > 0$ that for all $x \in K$ where K is some compact of space E_3 the inequality is fulfilled:

$$\left| \int_{E_3 - Ш_\rho} [\varphi_0(x-\beta t) - \varphi_0(x)] \mathcal{E}(t) dD_t \right| < \frac{\eta}{2} , \tag{27}$$

where $Ш_\rho$ is the sphere with radius ρ and the centre in the origin of coordinates.

By virtue of uniform continuity of function $\varphi_0(x)$ on K there exists such $\beta_1 > 0$ that for all $x \in K$ and $\beta \in (0, \beta_1)$ the inequality is valid:

$$|\varphi_0(x-\beta t) - \varphi_0(x)| < \frac{\eta}{2M} ,$$

where

$$M = \iint_{E_3} |\mathcal{E}(t)| dD_t \,.$$

Then

$$\left| \iint_{Ш_\rho} [\varphi_0(x - \beta t) - \varphi_0(x)] \mathcal{E}(t) dD_t \right| < \frac{\eta}{2}. \tag{28}$$

Summing (27) and (28) we obtain

$$\left| \int_{E_3} \mathcal{E}(x - \xi) \varphi_0(\xi) dD_\xi - \varphi_0(x) \right| =$$

$$\left| \iint_{E_3} [\varphi_0(x - \beta t) - \varphi_0(x)] \mathcal{E}(t) dD_t \right| < \eta. \tag{29}$$

for all $x \in K$. Taking into account (23) the last means that first of the conditions (2) is fulfilled. By analogy the second condition is proved.

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