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HOMOGENEOUS BY TIME MARKOV BRANCHING PROCESSES

Abstract

In the work for certain conditions of transient probabilities the homogeneous by time Markov branching processes are investigated.

In the present work we investigate the processes which are the propagation for the case of continuous time of determination of Galton-Vatson [1] processes. We will call such processes Markov branching processes.

These processes are special, but they represent an interest so as they reduce to more precise analytical results applicable to many cases of Galton-Vatson processes.

Now let us give the precise mathematical definition of Markov branching processes.

Let $\xi(t)$ be a random function, which represents the number of some population. This function receives non-negative entire values. If at some moment of time $\xi(t)=i$, then it is supposed that further history of population is determined by the law of distribution of probabilities which depends on i.

Let's denote

$$P_{ij}(\tau,t)=P\{\xi(t)=j|\xi(\tau)=i\},\ \ 0\le\tau\le t\,,\ \ i,\,j=0,1,2,\dots .$$

Definition. *Markov chain is named Markov branching process whose condition are entire non-negative numbers and whose transient probabilities are given by the solution of the system of differential equations*

$$\left.\begin{aligned}&\frac{\partial P_{ik}(\tau,t)}{\partial t}=-k\,b(t)P_{ik}(\tau,t)+b(t)\sum_{j=1}^{k+1}P_{ij}(\tau,t)\,j\,p_{k-j+1}(t)\\&P_{ik}(\tau,\tau+0)=\delta_{ik}\,,\end{aligned}\right\}\tag{1}$$

where $\delta_{ik}=1$ *for* $i=k$ *and* $\delta_{ik}=0$ *for* $i\ne k$.

It is supposed, that b is strong positive function, p_i are continuous and non-negative, $p_1(t)\equiv 0$ and $\sum\limits_{j=0}^{\infty}p_j(t)\equiv 1$.

If b and p_i don't depend on t then they say that the process is homogeneous by time.

Now let us obtain the equation for generating functions which is necessary at investigation of Markov branching processes.

Let $h(s,t)=\sum\limits_{k=0}^{\infty}p_k(t)s^k$, $|s|\le 1$ be a generating function of probabilities, and $F_i(s,\tau,t)=\sum\limits_{k=0}^{\infty}P_{ik}(\tau,t)s^k$, $|s|\le 1$ be a generating function of transient probabilities.

If we multiply both sides of the system (1) by s^k and sum by k, then we will obtain following equation

$$\frac{\partial F_i(s,\tau,t)}{\partial t}=b(t)[h(s,t)-s]\frac{\partial F_i(s,\tau,t)}{\partial s},\tag{2}$$

$$F_i(s,\tau,\tau+0)=s^i \quad , \qquad i=0,1,2,\dots .$$

As far as we consider the homogeneous by time Markov branching processes, then $h(s,t)=h(s)$ and $F_i(s,\tau,t)=F_i(s,t-\tau)$, and consequently equation (2) takes the form

$$\frac{\partial F_i(s,t)}{\partial t}=b[h(s)-s]\frac{\partial F_i(s,t)}{\partial s}. \tag{3}$$

Equation (3) is one of the important apparatuses for investigation of main characteristics of homogeneous by time Markov branching processes.

If in the inverse system to (1) we take $i=1$, multiply both sides by s^k and sum by k, then we obtain

$$\frac{\partial F_1}{\partial \tau}=-b(\tau)[h(F_1,\tau)-F_1], \tag{4}$$

$$F_1(s,t-0,t)=s\,, \quad t>0\,.$$

Further, because of that the considered processes are homogeneous by time Markov branching processes, then it is clear that b and h do not depend on t, probability P_{ij} and the generating functions F_i are the functions of argument $t-\tau$ and $\frac{\partial F_i}{\partial \tau}=-\frac{\partial F_i}{\partial t}$. Instead of $F_1(s,\tau,t+\tau)$, $h(s,t)$ we will write $F_1(s,t)$, $h(s)$. Then equation (4) has the form

$$\frac{\partial F_1(s,t)}{\partial t}=b[h(F_1(s,t))-F_1(s,t)],$$

$$F_1(s,0)=s\,, \quad t\geq 0\,.$$

Hence it follows that for $-1<s<1$

$$\int_s^{F_1(s,t)}\frac{dx}{h(x)-x}=b\cdot t\,.$$

The following theorem is valid:

Theorem. *For homogeneous by time Markov branching processes the generating function F_1, satisfies the correlation $F_1(1,t)\equiv 1$ then and only then when for any $\varepsilon>0$ the integral $\int_{1-\varepsilon}^{1}\frac{du}{h(u)-u}$ diverges.*

The proof is not given here.

Now let us establish the conditions for whose fulfillment the sum of transient probabilities is equal to unit.

If series $\sum k\,p_k(t)$ converges uniformly by t in each finite interval, then the only solution of system (1) satisfies identity $\sum_k P_{ik}(\tau,t)\equiv 1$.

Let t be fixed and $F_1(1,\tau,t)=G(\tau)$. Then from the differential equation (2) we have

$$\frac{dG}{d\tau}=-b(\tau)(h(G,\tau)-G),$$

where $G(t-0)=1$ and $0\leq G\leq 1$.

By virtue of the condition and definition of function h the right-hand side of this differential equation satisfies Lipshits's condition in the domain $-1\leq G\leq 1$, $0\leq\tau\leq t$.

Consequently, the solution is one and as far as 1 is the solution then $G(\tau)\equiv 1$. Hence $\sum_k p_{ik}(\tau,t)\equiv 1$.

Reference

[1]. Алиев С.А., Шуренко В.М. *Переходные явления и сходимость процессов Гальтона-Ватсона к процессам Иржины.* Теория вероятностей и ее применения. 1982, 27, №3, с.443-455.

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