DEHGHAN Y.N., SHAHBAZOV A.I.

WEIGHTED COMPOSITION TYPE OPERATORS ON THE SPACES OF VECTOR-VALUED FUNCTIONS

Abstract

In this paper we will investigate compactness of finite sums of weighted composition operators, acting on the spaces of vector-valued functions defined on a compact set X, namely on the cartesian products of uniformly closed subspaces of $C(X)$.

1. Introduction. Let X be a Hausdorff space and let $C(X)$ denote the space of all continuous complex-valued functions defined on X equipped with Sup-norm. Let $A(X)$ be a uniformly closed subspace of $C(X)$, and $C^n(X)$, $A^n(X)$ denote the cartesian products of $C(X), A(X)$ respectively, where $n \in N$. For any vector-valued function $u = (u^1,...,u^n)$ in $C^n(X)$ we introduce the following norm: $\|u\| = \max_{1 \leq i \leq n} \|u^i\|$, where

$$\|u^i\| = \max_{x \in X} |u^i(x)|, \ (i = 1,...,n).$$

The space $C^n(X)$ with this norm is a Banach space and $A^n(X)$ is a closed subspace of $C^n(X)$. Let $\omega_i : X \to X \ (i = 1,...,m)$ be continuous mappings. In this paper we will consider the operator $T : A^n(X) \to C^n(X)$ of the form $u(x) \mapsto \sum_{k=1}^{m} M_k(x) u(\omega_k(x))$, where $M_k(x) = (m_{ij}^k(x))_{i,j=1}^{n}$ is an $n \times n$ matrix of functions $m_{ij}^k \in C(X)$ for any k such that $1 \leq k \leq m$, and for any $i, j = 1,...,n$. In other words, we will consider the weighted composition operator $(m = 1)$ and its finite sums $(m > 1)$ on the uniformly closed subspaces $A^n(X)$ of continuous vector-valued functions on X. We will investigate compactness of the operator T. In the case $n = 1, m = 1$, i.e., compactness of weighted composition operator (w.c.o) on $C(X)$ ([9], [11]) and on its uniformly closed subspaces (in particular, on the uniform algebras) are studied in [2], [3], [4], [5], [8], [10], [11], [13]; and w.c.o. acting on the space of vector-valued continuos functions are studied in [14], [15], [16]. When $n = 1, m > 1$, i.e., compactness of the finite sums of weighted composition operators which are studied in [5], [6], [7], [12]. The importance of this kind of operators is that they are applicable in solving and the existence of solutions of equations of form $\sum_{k=1}^{m} M_k(x) u(\omega_k(x)) = f(x)$, i.e., the functional-differential equations containing both the argument and its shifts.

1. Representation Theorem. In this section we will find a relation between the compactness of $T : A^n(X) \to C^n(X)$ and the find finite sums of weighted composition operators acting on the space of complex-valued functions $A(X)$.

Let p_i be the projection $A^n(X) \to A(X)$, i.e., $p_i u(x) = u^i(x)$ where $u \in A^n(X)$, $u^i \in A_i(X)$ i-th factor of u, i.e., the projection to i-th factor $A_i(X)$; let $q_i : A(X) \to A^n(X)$ be the embedding for $A(X)$ into $A^n(X)$ as i-th factor. We define the operators $T_{ij} : A(X) \to C(X)$ and $T_i : A^n(X) \to C(X)$ such that $T_{ij} f(x) = \sum_{k=1}^m m_{ij}^k(x) f(\omega_k(x))$, for any $f \in A(X)$, $T_i u(x) = \sum_{j=1}^n T_{ij} u^j(x) = E \sum_{j=1}^n \left(\sum_{k=1}^m m_{ij}^k(x) u^j(\omega_k(x)) \right) = \sum_{k=1}^m \left(\sum_{j=1}^n m_{i,j}^k(x) u^j(\omega_k(x)) \right)$, for any $u \in A^n(X)$.

Now we can prove the following theorem, while assuming that $M_k(x) \neq 0$ (zero matrix) and $\omega_k(x) \neq constant$ for $k = 1,...,m$.

Theorem 2.1. *The operator* $T : A^n(X) \to C^n(X)$ *is compact if and only if, for any* $i,j = 1,...,n$ *the operators* $T_{ij} : A(X) \to C(X)$ *are compact.*

Proof. It is clear that $T_i = p_i \circ T$. Indeed, for any $u \in A^n(X)$

$$(p_i \circ T) u(x) = p_i \left(\sum_{k=1}^m M_k(x) u(\omega_k(x)) \right) =$$

$$= p_i \left(\sum_{k=1}^m \sum_{j=1}^n m_{1j}^k(x) u^j(\omega_k(x)), ..., \sum_{k=1}^m \sum_{j=1}^n m_{nj}^k(x) u^j(\omega_k(x)) \right) =$$

$$= \sum_{k=1}^m \sum_{j=1}^n m_{ij}^k(x) u^j(\omega_k(x)) = \sum_{j=1}^n \left(\sum_{k=1}^m m_{ij}(x) u^j(\omega_k(x)) \right) = \sum_{j=1}^n T_{ij} u^j(x) = T_i u(x),$$

and $T = \sum_{i=1}^n q_i \circ T_i$, indeed, for any $u \in A^n(X)$, $\sum_{i=1}^n q_i \circ T_i u(x) =$ $= \sum_{i=1}^n q_i \left(\sum_{k=1}^m M_k^i(x) u^i(\omega_k(x)) \right) = \sum_{k=1}^m M_k(x) u(\omega_k(x)) = T u(x)$, where denote i-th row of matrix $M_k(x)$.

Since the operators p_i and q_i are linear bounded ones, so T is compact if and only if T_i's are compact. On the other side, we can show that $T_i = \sum_{j=1}^n T_{ij} \circ p_i$ and $T_{ij} = T_i \circ q_j$. Indeed, $\sum_{j=1}^n (T_{ij} \circ p_i) u(x) = \sum_{j=1}^n T_{ij} u^j(x) = T_i u(x)$ for any $u \in A^n(X)$; and $(T_i \circ q_j) f(x) = T_i (0,...,f(x),...,0) = T_{ij} f(x)$ for any $f \in A(X)$. So the theorem is proved.

3. Compact Sums Of Weighted Composition Operators On Spaces Of Vector-Valued Functions.

In this section we will investigate the compactness of the operator T inducted by continuous mappings $\omega : X \to X$ $(i = 1,...,n)$ of the form $T : A^n(X) \to C^n(X)$, $u(x) \mapsto \sum_{k=1}^n M_k(x) u(\omega_k(x))$ as a simple deduction from Theorem 2.1 for certain uniform subspaces of $C^n(X)$ (in particular, for uniform sunalgebras).

Let $A(X)$ be a closed uniform subspace of $C(X)$ and $A^n(X) = A(X) \times ... \times A(X)$.

Definition 3.1. *A closed subset* $E \subset X$ *is called a peak set with respect to* $A^n(X)$, *if there exists a sequence* $\{u_k\}_{k=1}^\infty$ *such that* $u_k \in A^n(X)$, $|u_k^i(x)| = \|u_k^i\| = 1$ *for any* i $(1 \leq i \leq n)$, k *and* $x \in E$, *moreover, outside any neighborhood of* E *the sequence* $\{u_k^i\}$ *tends to* 0 *uniformly. A peak set consisting of only one point is called peak point.*

The set of all peak points with respect to $A^n(X)$ is denoted by Γ. Put $G = X \setminus \Gamma$. To each point $x \in X$ corresponds a functional $\delta_x : u \mapsto u(x)$ which lies in the

unit ball of the conjugate space $A(X)^*$. Thus induces the $A(X)^*$-topology, which is, generally speaking, stronger that the original one. Further, we shall always suppose that the original topology on G coincides with $A(X)^*$-topology, G is everywhere dense in X and Γ is not empty. A typical example is given by $A^n(D)$ where $A(D)$ is the disc algebra $\left(X = \overline{D}, \Gamma = \partial D = \overline{D} \setminus D, G = D\right)$. For such subspaces of $A^n(D)$ we consider the operator T induced by a finite number of mappings ω_i which preserve G, i.e., we assume that $\omega_i : X \to X$ are continuous mappings such that $\omega_i(G) \subset G, (i = 1,...,n)$. Except for easy degenerate case we shall assume that $\omega_i \neq constant$ for all i. As [6] we will identify the unit ball of the conjugate space $C(X)^*$ of $C(X)$ with all complex Borel measures on X with variation less than 1, any point $x \in X$ will correspond to a δ-measure. Since the compactness of an operator is equivalent to the compactness of its conjugate, so it can easily be shown that (see [1], Theorem VI, 7.1.) the compactness of the operators T_{ij} is equivalent to the continuity of the mappings $x \mapsto T_{ij}^* x = \sum_{k=1}^n M_{ij}^k(x)\omega_k(x)$ acting on X with original topology into $A(X)^*$ with metric topology. If $x \in G$ then because of $\omega_i(x) \in G$ and since the original topology coincides with $A(X)^*$-topology on G, then the above mentioned $x \mapsto T_{ij}^*(x)$ for $x \in G, (i, j = 1,...,n)$ is automatically continuous. This is the reason for the following definition.

Definition 3.2. *Let $\varsigma \in \Gamma$ be a fixed point, we say that indices i, j are equivalent with respect to $\varsigma \in \Gamma$ if $\omega_i(\varsigma) = \omega_j(\varsigma) \in \Gamma$. Equivalent classes will be denoted by K. K_0 will denote those indices i such that $\omega_i(\varsigma) \in G$. Indices i, j are called strongly equivalent if $\|\omega_i(x) - \omega_j(x)\|_{A(X)^*} \to 0$ when $x \to \varsigma$. Equivalent classes of this kind will be denoted by L.*

Now by using Theorem 2.1, Lemma 1 [7] the following theorem can easily be deduced.

Theorem 3.3. *If the operator $T : A^n(X) \to C^n(X)$ is compact then for any class K (with respect to ς) and for any $i, j = 1,...,n$ we have $\sum_{k \in K} m_{ij}^k(\varsigma) = 0$.*

The conditions $\omega_i(G) \subset G, i = 1,2...,m$ in Theorem 3.3 are essential as the following example (for the case $n = 1$) shows.

Example 3.4. *Let $X = \left\{z \in C : |z| \leq 1 \text{ or } |z - 2| \leq 1\right\}$, and let A be the algebra of all continuous functions on X and analytic inside, put $\omega_1(z) = z$ for all $z \in X$,*

$$\omega_2(z) = \begin{cases} z, & for \ |z| \leq 1, \\ 1, & for \ |z - 2| \leq 1, \end{cases} \qquad \omega_3(z) = \begin{cases} 1, & for \ |z| \leq 1, \\ z, & for \ |z - 2| \leq 1. \end{cases}$$

Since $\omega_i(1) = 1$ for all i, then for $\zeta = 1$ all indices are equivalent. Since $-f \circ \omega_1 + f \circ \omega_2 + f \circ \omega_3 = f(1)$, so that the operator is compact, but $m_{11}^1 + m_{11}^2 + m_{11}^3 = 1$.

Combinations of Theorem 2.1 and 1 [7] gives the following theorem.

Corollary 3.5. *The operator $T : A^n(X) \to C^n(x)$ is compact, if and only if, for an arbitrary point $\zeta \in \Gamma$ and for any $1 \leq i, j \leq n$ we have $\sum_{k \in K_0} m_{ij}^k(\zeta)\omega_k(x) \to 0$ with*

respect to $A(X)^*$ -norm as $x \to \zeta$ (in original topology of X) and $\sum_{k \in K} m_{ij}^k(\zeta) = 0$ for any class $K \neq K_0$.

Corollary 3.6. If $\sum_{k \in L} m_{ij}^k(\zeta) = 0$ for any class $1 \le i$, $j \le n$ and $\zeta \in \Gamma$, then the operator T is compact.

The following example (in the case $m = 1$) shows that the converse of the corollary is not in general true.

Example 3.7. Let X and A be as in Example 3.4. Put now:

$\omega_1(z) = z$ for all $z \in X$, $\omega_2(z) = \dfrac{z+1}{2}$ for all $z \in X$,

$$\omega_3(z) = \begin{cases} z, & \text{if } |z| \le 1, \\ \dfrac{z+1}{2}, & \text{if } |z-2| \le 1. \end{cases} \qquad \omega_4(z) = \begin{cases} \dfrac{z+1}{2}, & \text{if } |z| \le 1, \\ z, & \text{if } |z-2| \le 1. \end{cases}$$

It is obvious that $\omega_i(z) \in G$, when $z \in G$ and $\omega_i(1) = 1$ for all i, so for $\zeta = 1$ all indices are equivalent (consists of the class), however, we can easily show that no pairs of indices are strongly equivalent. But regardless of this, we have $-f \circ \omega_1 - f \circ \omega_2 + f \circ \omega_3 + f \circ \omega_4 = 0$, i.e., the operator corresponding to given ω_i and $m_{11}^i = \pm 1$ is zero.

But by using Theorem 4 [7] and Theorem 2.1, when $A(X)$ is uniform algebra and $m \le 3$ then the following theorem which is the converse of Corollary 3.6 can be proved.

Theorem 3.8. If $A(X)$ is a uniform algebra, then the operator $Tu(x) = \sum_{k=1}^m M_k(x) u(\omega_k(x))$ (where $m \le 3$), from $A^n(X)$ into $C^n(X)$ is compact if and only if, for any strong class L and for any $1 \le i$, $j \le n$ we have $\sum_{k \in L} m_{ij}^k(\zeta) = 0$.

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Dehghan Y.N.
Dept.of Mathematics, Tabriz University, Tabriz, Iran.

Shahbazov A.I.
Institute of Mathematics and Mechanics of AS Azerbaijan
9, F.Agayeva str., 370141, Baku, Azerbaijan.
Tabriz University, Tabriz, Iran.