# MATHEMATICS

## AKHUNDOV A.A.

## DEPENDENCE IN THE FORM OF SERIES BETWEEN DIFFERENTIAL INVARIANT OF QUASI-LINEAR HEAT EQUATION WITH ZERO AND ONE ORDERS

### Abstract

*The existence of a dependence between differential invariants with orders 0 and 1 in the form of the convergent row is proved. Coefficients of the row are defined by non-linear recurrent relations. Analysis of these relations allows to study qualitative properties of the initial non-linear equation. Particularly it has succeeded to obtain several exact solution of the initial equation.*

1. Let's consider the next quasi-linear equations:

$$u_t = k_0 \left( u^\sigma u_x \right)_x, \quad u(0,t) = \varepsilon(-t)^s, \quad u(x,-\infty) = 0 \tag{1}$$

and

$$u_t = -\widetilde{k}_0 \left( T^\mu(-t) \right)_t \left( u^\sigma u_x \right)_x, \quad u(0,t) = T(-t), \quad u(x,-\infty) = 0, \tag{2}$$

where $(x,t) \in [0,\infty) \times (-\infty,0)$, $k_0 > 0$, $\widetilde{k}_0 > 0$, $\sigma > 0$, $\varepsilon > 0$, $s < 0$, $\mu \neq 0$ are real parameters, $T(-t) \equiv R(-t) + \varepsilon(-t)^s$ is positive strictly increasing continuously differential function.

System (1) is automodel only at orders or exponential bounded regime. But system (2) is automodel in any permissible bounded regime [1], [2 for 1990].

System (1) and (2) coincides in fulfilling these conditions:

$$\mu s = 1, \quad \widetilde{k}_0 = k_0 \varepsilon^{-\mu}, \quad R(-t) \equiv 0.^1 \tag{3}$$

Automodel transformations:

$$\lambda = \widetilde{k}_0^{-1/2} T^{-(\sigma+\mu)/2}(-t)x,$$
$$u = T(-t)\phi^{1/(\sigma+1)}(\lambda) \tag{4}$$

allows reduce the equation (2) to the next form:

$$\phi^{\sigma/(\sigma+1)}\phi_{\lambda\lambda} - \frac{\sigma+\mu}{2\mu}\lambda\phi_\lambda + \frac{\sigma+1}{\mu}\phi = 0. \tag{5}$$

From relations (2) and (4) we get equality $\phi(0) = 1$. Connecting relations (3) and (4) allows to bound the investigate one of equations (1), (2) and (5).

At given work, after the short enumeration on the result of group analysis of differential equation (5), [1-3], taken dependence in form series between differential invariance with zero and one orders. The coefficients of series determine by non-linear recurrent relations. The theorem about convergence of this series is proved. The analysis of recurrent relations allowed in particularly to get exact solution of system (1), giving in work [4] and [5] when bounded regime of equation (1) is zero. In the result of analyses of

---

1 Refuses from the first equality in (3) allows to parameter $\mu$, except its analogues parameter $s$, can get not only negative and positive values too. In result community of system increases.

these recurrent relations we could get only unknown of non-trivial solution of equation (5) with zero begining condition $\phi(0) = 0$. The knowledge of possible high numbers of solutions allows to create more real representations about properties of studying non-linear processes and it is very important to avoid serious mistakes which begins in investigating of non-linear equations with approaching methods [3,5].

2. Equation (5) allows one-dimensional algebra Lie with infinitesimal operator $X = \dfrac{\sigma}{2(\sigma+1)} \lambda \dfrac{\partial}{\partial \lambda} + \phi \dfrac{\partial}{\partial \phi}$. To operator $X$ corresponds one parametric group of non-homogeneous tensions $\lambda' = \lambda \exp(\sigma a / 2(\sigma+1))$, $\phi' = \phi \exp a$. Operators:

$$\underset{1}{X} = X + \frac{\sigma+2}{2(\sigma+1)} \dot{\phi} \frac{\partial}{\partial \dot{\phi}}, \quad \underset{2}{X} = \underset{1}{X} + \frac{1}{\sigma+1} \ddot{\phi} \frac{\partial}{\partial \ddot{\phi}}$$

are continuos of operator $X$ to the first and second derivatives. The solution of equality $\underset{2}{X} J = 0$ invariants including second order $U = \phi \lambda^{-2(\sigma+1)/\sigma}$, $V = \dot{\phi} \lambda^{-(\sigma+2)/\sigma}$, $W = \ddot{\phi} \lambda^{-2/\sigma}$. The differential invariant of the second order can be get by the Lie theorem with differentiating $DV / DU$. This conformation allows to put down the order of equation to one it passing from equation (5) to the correspondingly equation in differential invariant.

$$\left( V - \frac{2(\sigma+1)}{\sigma} U \right) \frac{dV}{dU} = \left( \frac{\sigma+\mu}{2\mu} U^{-\sigma/(\sigma+1)} - \frac{\sigma+2}{\sigma} \right) V - \frac{\sigma+1}{\mu} U^{1/(\sigma+1)}. \tag{6}$$

3. The assertion is truth.

**Theorem 1.** *In fulfilling condition* $\phi(0) = 1$ *between differential invariant of the first order* $V$ *and zero order* $U$ *exist the dependence in the form of series*

$$V = \sum_{i=0}^{\infty} a_i U^{v_i}, \tag{7}$$

*where fractional power* $v_i = (1 - i\sigma)/(1 + \sigma)$, *but coefficients* $a_i$ *are determined by non-linear recurrent relations*

$$a_{i+1} = \sum_{k=0}^{[i/2]} L_i^k a_{i-k} a_k, \quad i = 0,1,2,... \tag{8}$$

*with initial condition* $a_0 = -(\sigma+1)/\mu$.

Here

$$L_i^k = \begin{cases} (\mu - (2i-1)\sigma + 4\alpha_i^0)/2\mu(2i+3)a_0, & k = 0, \\ (i\sigma - 2)\alpha_{i-k}^k /(1+\sigma)(2i+3), & k = 1,2,...,[i/2]; \end{cases}$$

$$\alpha_i^j = \begin{cases} 1, & i \neq j, \\ 1/2, & i = j. \end{cases}$$

The proof of the theorem is based on some non-traditional application of a method of undetermined coefficients. In case of the non-linear equations example: form (6), immediate application of a method of undetermined coefficients, assumed searching of a solution of the equation in form of usual power series with the following equate of coefficients gives nothing. However, at the proof of the theorem 1 clarified that if by foreseen replace the power series by fraction power series of the type (7) as with unknown coefficients as with unknown powers, then after such modification the method is applied successfully and in the case of non-linear equation (6), which is as known the special form of Abel's equation of the second kind. The last means that the theorem 1 has also independent value. In connection with the told and also with assertion of theorem 1,

the problem of definition of conditions of convergence of the series (7) acquires special importance. The following point is devoted to the solution of this problem

4. Using the values of invariant from p.2 the series (7) can be represented in the form

$$\dot{\phi}\lambda^{-1-2/\sigma} = \sum_{i=0}^{\infty} a_i \left(\phi\lambda^{-2(\sigma+1/\sigma)}\right)^{\nu_i},$$

from where after passing to the new variable $\tau$, determined by equality $X\tau = 1$, we get equation

$$U_\tau + U - \frac{\sigma}{2(\sigma+1)}\sum_{i=0}^{\infty} a_i U^{\nu_i} = 0 \tag{9}$$

belonging to the class of equations with dividing variables.

Transformation of the type Bernoulli

$$\Lambda = U^{1-\nu_0} \tag{10}$$

reduces the equation (9) to the next form

$$\Lambda_\tau + (1+\nu_0)\Lambda - \frac{(1-\nu_0)^2}{2}\sum_{i=0}^{\infty} a_i \Lambda^{-i} = 0 \tag{11}$$

Remember that the series:

$$g(\lambda) = \sum_{i=0}^{\infty} a_i \Lambda^{-i} \tag{12}$$

in the case when ordinary real variable is used instead of invariant is called either asymptotic series [6, p.531-551] or power asymptotic series [7, p.133]. Utilization of transformation (10) is truth, as for as $\Lambda$, obviously, is invariant of zero order.

According to the relations (9)-(11) from conditions of convergence of the series (12) we can get convergence of the series (7) and vice versa. This observation appears useful after completion of the proof of the following assertion.

**Theorem 2.** *Let the number* $K_0 = K_0(\mu,\sigma)$ *at any* $\sigma, \sigma > 0$ *and* $\mu, \mu \neq 0$ *are defined by equality*

$$12(\sigma+1)K_0 = \begin{cases} \max\{4, -(\mu+\sigma+2)\}, & \mu \leq -4(\sigma+1), \\ \max\{4, \mu+7\sigma+6\}, & \mu > -4(\sigma+1), \end{cases} \tag{13}$$

*Then at fulfilling the equalities*

$$\omega \leq \Lambda < +\infty, \tag{14}$$

*where* $\omega = 4(\sigma+1)K_0/|\mu|$, *the series (12) is convergent and is bounded by the sum*

$$\Lambda\left(1 - (1-\omega/\Lambda)^{1/2}\right)/2K_0 \tag{15}$$

*by the mojorant series*

$$h(\Lambda) = \sum_{n=0}^{\infty} b_n \Lambda^{-n}. \tag{16}$$

*Here* $b_n = (\omega/4)^{n+1} C_n/K_0$, $C_n = \dfrac{1}{n+1}C_{2n}^n$ *are Catalon numbers.*

**Proof.** Taking into account recurrent relation (8) let's represent the series (12) in the following form:

$$g(\Lambda) = a_0 + \Lambda^{-1}\sum_{n=0}^{\infty}\left[2p_n(\mu)a_n a_0 + q_n 2\sum_{k=1}^{[i/2]} \alpha_{n-k}^k a_{n-k} a_k\right]\Lambda^{-n} \tag{17}$$

where

$$p_n(\mu) = \frac{n\sigma - 2\alpha_n^0 - (\mu + \sigma)/2}{2(\sigma + 1)(2n + 3)}, \quad q_n = \frac{n\sigma - 2}{2(\sigma + 1)(2n + 3)}.$$

Obviously, that $\lim_{n \to \infty} p_n(\mu) = \lim_{n \to \infty} q_n = \sigma/4(\sigma + 1)$. The sequence $\{q_n\}$ is strictly monotonously increasing. For other sequence (without first element) one of the following variants is possible:

1) If $\mu > -4(\sigma + 1)$, then the sequence $p_1(\mu), p_2(\mu),...$ is strictly monotonously increasing and is bounded from above by limit $\sigma/4(\sigma + 1)$.

2) If $\mu < -4(\sigma + 1)$, then the sequence $p_1(\mu), p_2(\mu),...$ is strictly monotonously decreasing and is bounded from below by limit $\sigma/4(\sigma + 1)$.

3) If $\mu = -4(\sigma + 1)$, then $p_1(\mu), p_2(\mu) = ... = \sigma/4(\sigma + 1)$.

The analysis of variants 1-3, with obligatory consideration of the first element $p_0(\mu)$ sequence $\{p_n(\mu)\}$, shows that number $K_0 = K_0(\mu, \sigma)$ determined by equality (13) majorites all elements of the sequences $\{q_n\}$ and $\{p_n(\mu)\}$. Let's note that $K_0 \geq 1/3(\sigma + 1)$ for all $\sigma > 0$, $\mu \neq 0$ define the sequence of positive numbers $\{b_n\}$ with recurrent relations

$$b_{n+1} = 2K_0 \sum_{k=0}^{[n/2]} \alpha_{n-k}^k b_{n-k} b_k, \quad b_0 = |a_0|, \quad n = 0,1,2,.... \tag{18}$$

Using the formula (17) we can prove the validity of inequalities $|a_n| \leq b_n$, $n = 0,1,2,...$, by the method of mathematical induction, meaning that the series (16) majorites the series (12). The convergence of the series (16) can be determined by method of producing functions allowing to calculate its sum explicitly. Let's consider one more series of form

$$f(\Lambda) = \sum_{n=0}^{\infty} c_n \Lambda^{-n}, \tag{19}$$

where $c_n = K_0 b_n$, $n = 0,1,2,...$. However, $f(\Lambda) = K_0 h(\Lambda)$ then series (16) and (19) can convergence or divergence simultaneously. Requrent relations for coefficients $c_n$ are obtained from equalities (18) by the way of multiplying of both sides by multiplier $K_0$:

$$c_{n+1} = 2 \sum_{k=0}^{[n/2]} \alpha_{n-k}^k c_{n-k} c_k, \quad c_0 = K_0 |a_0| \tag{20}$$

Thus, as a result of introduction of a series (19) arises the problem distinguishes from corresponding combinator problem [7, p.137-139] of non-uniqueness of initial value $c_0$. Indeed we'll consider the series

$$C(s) = \sum_{n=0}^{\infty} C_n s^n, \tag{21}$$

which the coefficients determined by the recurrent relations

$$C_{n+1} = 2 \sum_{k=0}^{[n/2]} \alpha_{n-k}^k c_{n-k} c_k, \quad C_0 = 1^1 \tag{22}$$

---

[1] Of course, relations (22) can written in equivalent form, used in work [7, p.138]. But, we must agree that the note (22) is more perfectly mathematically, as far as in grouping such elements are connected least (practically half) elements in it.

Next auxiliary assertion proved by method of mathematical induction, transform the connection between the relations (20) and (22).

**Lemma 1.** *Between the coefficients of the series (19) and (21) exist the connection, defined by equalities* $c_n = C_n c_0^{n+1}$, $n = 0,1,2,\ldots$.

From lemma 1 and formulas (16), (18)-(22) we get equality:

$$h(\Lambda) = \frac{\sigma + 1}{|\mu| C(c_0 / \Lambda)}.$$  (23)

According to [7], the sequence $\{C_n\}$ which defined by recurrent relations (22) consists of number Catalon, i.e. $C_n = \frac{1}{n+1} C_{2n}^n$, $n = 0,1,2,\ldots$. Thus this series (21) is transformation function of sequence $\{C_n\}$ and has the sum

$$C(s) = (1 - \sqrt{1 - 4s}) / 2s.$$

Taking into account the limit $\lim_{s \to 0} C(s) = 1$ and the relation $h(\Lambda) = |h(\Lambda)|$ we get that at fulfilling the condition $s \in (0,1/4]$ the transformation $s = c_0 / \Lambda$ in formula (21) is truth. From which, after the account of value $c_0$, follows the inequalities (14). The formula (15) follows from formula (23). Theorem is proved.

From the theorem 2 and from consideration expounded before its formulation, we get the assertion about convergence of fraction-power series (7).

**Theorem 3.** *The series (7), which coefficients and fraction powers determined correspondingly by recurrent relations. (8) and by equalities* $v_i = (1 - i\sigma)/(1 + \sigma)$, $i = 0,1,2,\ldots$, *are convergence at* $U \in [\omega^{1+1/\sigma}, +\infty)$ *bounded by sum*

$$U\left(1 - (1 - \omega U^{v_0 - 1})^{1/2}\right) / 2K_0$$

*of majorant series* $\sum_{i=0} b_i U^{v_i}$.

5. The assertion is truth.

**Theorem 4.** *At fulfilling the condition* $\phi(0) = 0$ *between the differential invariants of the first order* $V$ *and the zero order* $U$ *exists the dependence in form of series*

$$V = \sum_{i=0}^{\infty} a_i^0 U^{v_i - 1},$$  (24)

*where coefficients* $a_i^0$ *is determined by recurrent relations*:

$$a_i^0 = \begin{cases} A_i a_{i-1}^0 + B_i \sum_{k=2}^{[i/2]} a_k^0 a_{i-k}^0, & i = 2n + 1, \\ A_i a_{i-1}^0 + B_i \sum_{k=2}^{(i/2)-1} a_k^0 a_{i-k}^0 + \frac{1}{2} B_i (a_{i/2}^0)^2, & i = 2(n+1), \end{cases}$$  (25)

$a_0^0 = 1$, $a_1^0 = \frac{(\sigma + 1)(\mu - \sigma - 2)}{2\mu(2\sigma + 3)}$, $a_2^0 = \frac{(\sigma + 1)^2(\mu + \sigma + 1)}{\mu(2\sigma + 3)(3\sigma + 5)} a_1^0$.

*Here* $n$ *gets the value from the set of natural numbers,*

$$A_i = \frac{(\sigma + 1)(\mu(1 + i\sigma) + (4 - i)\sigma^2 + (9 - 2i)\sigma + 4)}{2\mu(2\sigma + 3)((i+1)\sigma + 2i + 1)},$$

$$B_i = \frac{(i-2)\sigma - 2}{(i+1)\sigma + 2i + 1}, \qquad \sum_{k=m_1}^{m_2\circ} f_k = \begin{cases} \sum_{k=m_1}^{m_2} f_k, & m_2 \geq m_1, \\ 0, & m_2 < m_1. \end{cases}$$

The proof of theorem 4 is similar to the proof of the theorem 1, i.e. transform by applying of modified variant of the method of undetermined coefficients.

Basing on connections, which exists between considerable fraction-powers and powers asymptotic series it is not difficult to prove the uniqueness of both representations in form of series (7) and (24), got at initial values $\phi(0) = 1$ and $\phi(0) = 0$ correspondingly. As for as the opposite assertion is incorrect for power asymptotic series [6, p.536], then it is incorrect for fraction-power series (7) and (24).

In p. 2 was denoted, that equation (5) suppose one-parametric group of non-homogeneous tensions. It is interesting that with recurrent relations (8) have analogical property. The assertion is truth

**Lemma 2.** *The recurrent relations (8) suppose one-parametric group non-homogeneous tensions* $a'_i = a_i \exp((i+1)a)$, *with infinitesimal operator*

$$Y = \sum_{n=0}^{\infty} (n+1)a_n \frac{\partial}{\partial a_n}, \quad i = 0,1,2,\ldots$$

Lemma 2 doesn't keep the power for recurrent relations (25). It means that between recurrent relations (8) and (25) exists the qualitative difference.

In result, let count some of the results, got from the representations (7), (8) and (23), (24). At $\mu = -(\sigma + 2)$ from (7) and (8) we conclude the exact solution of system (1), giving in work [5]. This solution, as known, brings real correctives in understanding of physics, described by the system (1). At $\mu = -\sigma$ from analysis of relations (7), (8) follows exact solution of system (1), given in work [4]. With this we could get previously unknown exact solution of system (1) [8]. Obviously, every of these solutions allows to get exact solution from any of the equations (2), (5) and (6). Another previously unknown solution of equation (5), satisfied the condition $\phi(0) = 0$ has the text from

$$\phi(\lambda) = \left( c\lambda^{\frac{\sigma}{\sigma+1}} + \frac{\sigma}{2(\sigma+2)}\lambda^2 \right)^{\frac{\sigma+1}{\sigma}}$$

and gets from representations (24), (25) for condition $\mu = -(\sigma + 2)$, where $c$ is real constant.

## References

[1]. Ахундов А.А. *Нелинейное уравнение теплопереноса - автомодельное в классе дифференцируемых граничных режимов*. Тезисы докл. Всесоюзн. Конф. «Негладкий анализ и его приложения к мат. Экономике», Баку, 30 сентября – 6 октября, 1991, с.91.

[2]. Ахундов А.А. В ежегодных отчетах сектора 1.1 ИК АН Азербайджана за 1990-1998 гг.

[3]. Ахундов А.А. *Явление локализации энергии и исследование нелинейных уравнений методом универсальной автомодельности*. II Международный Симпозиум «Проблемы мат. Моделирования, управления и информационных технологий в нефтегазовой промышленности», Баку, 21-26 сент., Известия АН Азербайджана, сер. ф.-т. и матем. н., 1998 г., т.XVIII, №1, с.210-212.

[4]. Самарский А.А., Змитренко Н.В., Курдюмов С.П., Михайлов А.П. *Эффект метастабильной локализации тепла в среде с нелинейной теплопроводностью*. ДАН СССР, т.223, №6, 1975, с.1344-1347.

[5]. Ахундов А.А. *О необходимости доисследования одного квазилинейного уравнения теплопроводности*. Известия АН Азербайджана, сер. ф.-т. и матем. н., 1993, №5-6, с.40-42.

[6]. Фихтенгольц Г.М. *Курс дифференциального и интегрального исчисления*. М., Наука, 1969, т.II, 800 с.

[7]. Сачков В.Н. *Введение в комбинаторное методы дискретной математики*. М., Наука, 1982 г., 384 с.

[8]. Achundov A.A. *The impossibility of energy localization the processes with sharpening described by quasi-linear equation of heat-transfer*. Second international symposium on mathematical and computational applications. Baku, September 1-3, 1999, p.54.

**Akhundov A.A.**
Institute Cybernetics of AS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-09-42(off.), 38-20-06(apt.).