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TO THE SPECTRAL ANALYSIS OF ORDINARY DIFFERENTIAL OPERATORS POLYNOMIALLY DEPENDING ON A SPECTRAL PARAMETER WITH PERIODIC MATRIX COEFFICIENTS

Abstract

In the paper the complete spectral analysis of the operators is carried out and also with help of the generalized normalizing matrices the inverse problem is solved.

Some results of paper [3] obtained in the scalar differential bundle case are generalized for a differential bundles system generated by differential expressions

$$l(Y) \equiv (-1)^m Y^{(2m)}(x) + \sum_{\gamma=0}^{2m-2} P_\gamma(x,k) Y^{(\gamma)}(x) \tag{1}$$

where $y(x) = (y_1(x), \dots, y_m(x))$ are vector-functions from $L_2^m(-\infty,\infty)$ with m components from $L_2(-\infty,\infty)$. We assume that matrix coefficients have the special form

$$P_\gamma(x,k) = \sum_{s=0}^{2m-\gamma-2} k^s \sum_{n=1}^{\infty} \{P_{\gamma sn}\} e^{inx} \tag{2}$$

where $P_{\gamma sn}$ is a quadratic matrix of order m, whose elements belong to numerical set and the series

$$\sum_{\gamma=0}^{2m-2} \sum_{t=0}^{2m-\gamma-2} \sum_{s=1}^{\infty} s^{\gamma+t+1} \|P_{\gamma ts}\| = P \tag{3}$$

converges.

We see from expression (2) that $P_\gamma(x,k)$ is a 2π periodic matrix function and it admits an analytic continuation to upper half-plane.

This case allows to conduct analysis of the operator L. It turns out to be that a spectrum of the operator L is continuos and it fills the axis

$$\left\{ k\,\omega_j \middle| -\infty < k < \infty,\ j = \overline{0,2m-1},\ \omega_j = \exp\left(i\,j\,\pi/m\right) \right\},$$

and there are spectral properties on the continuos spectrum in the sense of M.A.Naimark [1], that coincide with numbers of the form $n\omega_j/2$, $j = \overline{0,2m-1}$, $n = \pm 1, \pm 2, \dots$. Later, the inverse problem for reestablishment of coefficients of matrix functions $P_{js}(x)$ is solved on generalized normalizing matrices.

Note that second order periodic operator with potential $p(x) = \sum_{n=1}^{\infty} q_n e^{inx}$, where q_n is the order quadratic matrix has been studied by M.G.Gasymov and others [2].

1. Special solutions of the equation $l(Y) = k^{2m} Y$. Intriduce some notations. Let

$$\omega_j = \exp\left(i\pi j/m\right) : j = \overline{0,2m-1}$$

$$k_{nj\tau} = -n\omega_\tau^{-1}(1-\omega_j)^{-1} : j = \overline{0,2m-1},\ \tau = \overline{1,2m-1},$$

$$\frac{1}{n+k\omega_\tau(1-\omega_j)}\left[(\alpha+k\omega_\tau)^{2m}-k^{2m}-(\alpha+k_{nj\tau}\omega_\tau)^{2m}+k_{nj\tau}^{2m}\right]=$$

$$=\sum_{\gamma=0}^{2m-2}C_{j\tau\gamma}(n,\alpha)k^\gamma : j=\overline{1,2m-1},\ \tau=\overline{0,2m-1},\ n\in N,\ n<\alpha$$

$$\frac{k^s(r+k\omega_\tau)^v-k_{nj\tau}^s(r+k_{nj\tau}\omega_\tau)^v}{k\omega_\tau(1-\omega_j)+n}=\sum_{\gamma=0}^{v-1}C_{j\tau\gamma s}(n,r,v)k^{\gamma+s} \tag{4}$$

Theorem 1. *Let the conditions (2) and (3) be fulfilled. Then a differential equation $l(Y)=k^{2m}Y$ has matrix solutions $f_j(x,k)$ representable as*

$$f_\tau(x,k)=\exp(ik\omega_\tau x)\left[I+\sum_{j=1}^{2m-1}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\left\{\upsilon_{n\alpha}^{(j,\tau)}\right\}}{n+k\omega_\tau(1-\omega_j)}\exp i\alpha x\right], \tag{5}$$

where $\left\{\upsilon_{n\alpha}^{(j,\tau)}\right\}$ are m order matrices., for $\gamma=\overline{0,2m-2}$, $\alpha=1,2,\ldots$ and for $\tau=\overline{0,2m-1}$ and

$$\sum_{j=1}^{2m-1}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n+1}^{\infty}\alpha^{2m-1}(\alpha-n)\left\|\upsilon_{n\alpha}^{(j,\tau)}\right\|; \tag{6}$$

$$\sum_{j=1}^{2m-1}\sum_{n=1}^{\infty}n^{2m-1}\left\|\upsilon_{nn}^{(j,\tau)}\right\| \tag{7}$$

converge.

The proof of the theorem is obtained from the assumption that the series (5) admits term-by-term differentiation $2m$ times, and therefore we can directly substitute this series into equation. By equating the coefficients for $\exp(i\alpha x)$ we get identities with rational functions that have poles at the point $k_{nj\tau}$, $n=\overline{1,2m-1}$, $j=\overline{0,2m-1}$, $\tau=\overline{1,2m-1}$, and the sum of their reduses at these points must vanish. If we take into consideration all these facts, we get a matrix equations system for the definition of $\left\{v_{n\alpha}^{j\tau}\right\}$ by $\{p_{\gamma sn}\}$.

$$\begin{gathered}\left[\left(\alpha-\frac{n}{1-\omega_j}\right)^{2m}-\left(\frac{n}{1-\omega_j}\right)^{2m}\right]\left\{\upsilon_{n\alpha}^{(j,\tau)}\right\}+\\+\sum_{\gamma=0}^{2m-2}\sum_{s=0}^{2m-\gamma-2}\sum_{r+t=\alpha}i^\gamma\left(r-\frac{n}{1-\omega_j}\right)^\gamma\left(\frac{n}{\omega_\tau(1-\omega_j)}\right)^s\{P_{\gamma st}\}\left\{\upsilon_{nr}^{(j,\tau)}\right\}=0\end{gathered} \tag{8}$$

for $\alpha=2,3,\ldots$; $\gamma=\overline{1,2m-1}$, $n\in N$; $n<\alpha$

$$\begin{gathered}\sum_{s=0}^{2m-\gamma-2}\{P_{\gamma s\alpha}\}i^\gamma\omega_\tau^\gamma+\sum_{j=1}^{2m-1}\sum_{n=1}^{\alpha}C_{j\tau\gamma}(n,\alpha)\left\{\upsilon_{n\alpha}^{(j,\tau)}\right\}+\\+\sum_{j=1}^{2m-1}\sum_{v=\gamma}^{2m-2}\sum_{s=0}^{2m-v-2}\sum_{r+t=\alpha}\sum_{n=1}^{r}i^v\{P_{vst}\}C_{j\tau\gamma s}(n,r,v)\left\{\upsilon_{nr}^{(j,\tau)}\right\}=0\end{gathered} \tag{9}$$

This system is solved recurrently: the matrix $\left\{v_{n\alpha}^{(j,\tau)}\right\}$ whose elements are expressed by the elements of the matrix $\{p_{jsn}\}$ is directly determined.

Corollary 1. *The matrix functions* $f_s(x,k)$, $s=\overline{0,2m-1}$ *form a fundamental system of solutions of the equation* $l(Y)=k^{2m}Y$ *for* $k\neq 0$, $k\neq k_{njs}$, $n\in N$, $j=\overline{1,2m-1}$, $s=\overline{0,2m-1}$.

Divide the complex k-plane into $2m$ equal sectors S_v determined by the inequality $S_v=\left\{\frac{v\pi}{m}<\arg k<\frac{(v+1)\pi}{m},\ v=\overline{0,2m-1}\right\}$, at each sector S_v we can choose such arrangement of numbers ω_v that for $k\in S_v$ it is fulfilled the inequality

$$\mathrm{Re}(k\omega_0)\leq\ldots\leq\mathrm{Re}(k\omega_{m-1})<0<\mathrm{Re}(k\omega_m)\leq\ldots\leq\mathrm{Re}(k\omega_{2m-1}).$$

If $k\neq k_{njs}$, $k\in S_v$ then $\|f_s(x,k)\|\in L_2(0,\infty)$, $s=\overline{0,m-1}$, and $\|f_s(x,k)\|\in L_2(-\infty,0)$, $s=\overline{m,2m-1}$.

It is obvious that $f_s(x,k)$ may have the poles of the first order at the point $k_{njs}=-{n}/{\omega_s(1-\omega_j)}$. Introduce the notation

$$f_{nj}^{s}(x)=\lim_{k\to k_{njs}}\left[n+k\omega_s(1-\omega_j)\right]f_s(x,k) \tag{10}$$

Corollary 2. *For any values of* n and $j,s=\overline{0,2m-1}$, it holds the equality

$$\begin{gathered} f_{nj}^{s}(x)=\left\{\upsilon_{nn}^{(j,s)}\right\}f_{j+s}(x,k_{njs}) \\ f_{2m-s}(x)\equiv f_s(x,k),\ s=\overline{0,2m-1} \end{gathered} \tag{11}$$

2. Spectral solutions of a transposed equation. A system of equations

$$(-1)^m z^{(2m)}(x)+\sum_{\gamma=0}^{2m-2}(-1)^m\left[P_\gamma(x,k)z(x)\right]^{(\gamma)}=k^{2m}z(x) \tag{12}$$

is obtained by transposing the system of equations $l(Y)=k^{2m}Y$.

We can easily check that in these equations the coefficients at the derivative also satisfy the conditions (2), (3). Therefore the equation (12) also has a matrix solution $\varphi_s(x,k)$ representable in the view

$$\varphi_s(x,k)=\exp(-ik\omega_s x)\left[I+\sum_{j=1}^{2m-1}\sum_{n=1}^{\infty}\frac{1}{n-k\omega_s(1-\omega_j)}\sum_{\alpha=n}^{\infty}\left\{R_{n\alpha}^{(j,s)}\right\}\exp i\alpha x\right] \tag{13}$$

and the series of (6) and (7) type by the change of $\left\{\upsilon_{n\alpha}^{(j,s)}\right\}$ to $\left\{R_{n\alpha}^{(j,s)}\right\}$ converge.

3. On the resolvent of the operator L.

Theorem 2. *At any value of* k *from the sector* S_v, $k\neq k_{njs}$ *the resolvent* $R_k=\left(L-k^{2m}E\right)^{-1}$ *exists, it is a bounded integral operator in* $L_2^m(-\infty,\infty)$ *and it is generated by the kernel*

$$R(x,t,k)=-\frac{i}{2mk^{2m-1}}\begin{cases}\sum\limits_{s=0}^{m-1}\omega_s f_s(x,k)\varphi_s(x,k), & t<x\\ \sum\limits_{s=m}^{2m-1}\omega_s f_s(x,k)\varphi_s(x,k), & t>x\end{cases} \tag{14}$$

This representation admits to study a spectrum of the operator L.

Theorem 3. *The operator bunch* L_k *has a pure continuous spectrum that fills the axis* $\{k\omega_j | -\infty < k < \infty; j=\overline{0,2m-1},\ \omega_j = \exp(i\pi/m)\}$, *and on the continuous spectrum may be spectral properties that coincide with the numbers of the form* $n\omega_j/2$, $n = \pm 1, \pm 2, \ldots$

Theorem 4. *For any* $k \in S_\nu$, *it is valid the representation for the kernel of the matrix*

$$R(x,t,z) = \sum_{\nu=0}^{2m-1}\sum_{n=1}^{\infty} \frac{i\{\upsilon_{nn}^{(m,\nu)}\}}{2m\left(n\frac{\omega_\nu}{2} - z\right)\left(\frac{n}{2}\right)^{2m-1}\omega_\nu^{2m-2}} f_\nu\left(x, \frac{n\omega_\nu}{2}\right)\varphi_{m+\nu}\left(t, \frac{n\omega_\nu}{2}\right) -$$

$$-\frac{1}{2\pi i}\sum_{\nu=0}^{2m-1}\int_{\Gamma_\nu^-}^{\omega} \frac{\omega_{2m-\nu}f_{2m-\nu}(x,k)\varphi_{2m-\nu}(t,k) - \omega_{m-\nu}f_{m-\nu}(x,k)\varphi_{m-\nu}(t,k)}{k-z}dk \qquad (15)$$

Γ_ν^- *is obtained from* Γ_0^- *(the contour formed by subsegments* $[0, 1/2 - \delta]$, $[n/2 + \delta, (n+1)/2 - \delta]$, $n = 1,2,\ldots$ *and semicicles of radius* δ *with centers at points* $n/2$, $n = 1,2,\ldots$ *arranged at the low half-plane).*

In the formula (15) the numbers $\{\upsilon_{nn}^{(m,\nu)}\} = \{S_{nm\nu}\}$ play the role of normalizing matrix functions responding to spectral properties.

The inverse problem: First we get evident relations between the matrices $\{S_{nm\nu}\}$ and $\{\upsilon_{n\alpha}^{(m,\nu)}\}$. For this we use the identity (11); we find that

$$\{\upsilon_{nn}^{(j,\nu)}\} = \{S_{nj\nu}\} \qquad (16)$$

$$\{\upsilon_{n,\alpha+n}^{(j,\nu)}\} = \{S_{nj\nu}\}(1-\omega j)\sum_{s=1}^{2m-1}\sum_{r=1}^{\alpha} \frac{\{\upsilon_{r\alpha}^{(s,j+\nu)}\}}{r(1-\omega_j) - n\omega_j(1-\omega_s)} \qquad (17)$$

for $j = 1,2,\ldots,2m-1$, $\alpha, n = 1,2\ldots$. These relations are basic equations to determine $\{P_{\gamma n}\}$ on $\{S_{nj\nu}\}$.

In fact, if the normalizing matrices $\{S_{nj\nu}\}$ are known, then (16) and (17) give recurrent formulas to define the matrices $\{\upsilon_{nn}^{(m,\nu)}\}$. Then from (9) the numbers $\{P_{\gamma n}\}$ are defined uniquely.

The proof of theorem 2 is similar to the proof carried out in paper [3].

Thus, the inverse problem has a unique solution, and the numbers $\{P_{\gamma n}\}$ are effectively determined by normalizing matrices.

There arises a question: when does the given totality of normalizing matrices $S_{nj\nu}$, $n = 1,2,\ldots$, $j = 1,2,\ldots,2m-1$, $\nu = 0,\ldots,2m-1$ coincide with totality of normalizing matrices of L type operator?

To formulate the answer we introduce the following denotations:

$$a_m = \max_{\substack{1\le j \le l \le 2m-1 \\ 1 \le n, r < \infty}} \frac{|(1-\omega_j)(n+2)|}{|r(1-\omega_j) - n(1-\omega_l)\omega_j|} \qquad (18)$$

$$\{S_n\} = \sum_{\nu=0}^{2m-1}\sum_{j=1}^{2m-1} n^{2m-2}\|S_{nj\nu}\|. \qquad (19)$$

Theorem 5. *Let for the given matrices* *the following conditions be fulfilled*

$$I.\ \sum_{n=1}^{i} n\|S_n\| < \infty \tag{20}$$

$$II.\ 4^{m-1} a_m \sum_{n=1}^{\infty} \frac{\|S_n\|}{n+1} = p < 1 \tag{21}$$

Then there exists the functions $P_\gamma(x,k)$, $\gamma = \overline{0,2m-2}$ *of the form (2), for which the condition (3) is fulfilled, and the matrices* $\{S_{nj\nu}\}$ *are "generalized normalizing matrices" of the operator with reestablished coefficients.*

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