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ON MULTIPLE COMPLETENESS OF A PART OF ROOT VECTORS OF A CLASS OF POLYNOMIAL OPERATOR BUNDLES

Abstract

In the work the sufficient conditions are obtained which ensure multiple completeness of the part of root vectors responding to eigen values from the left half-plane of a class of polynomial operator bundles with normal main part.

In Hilbert separable space H let us consider the polynomial operator bundle of $n=2k$ -th order

$$P(\lambda)=(-1)^k\lambda^n E+\lambda^{n-1}A_1+\cdots+\lambda A_n+A^n, \tag{1}$$

where coefficients of bundle (1) satisfy following conditions:

1) spectrum of operator A is contained in finite number of rays from the angular sector

$$S_\varepsilon=\{\lambda\,|\,|\arg\lambda|\le\varepsilon\}\ ,0\le\varepsilon<\pi/n\,;$$

2) there exists quite continuous inverse A^{-1}, i.e. $A^{-1}\in\sigma_\infty(H)$;
3) operators $B_j=A_j\cdot A^{-j}$ $(j=1,\ldots,n)$ are bounded in H, i.e. $B_j\in L(H)$;
4) operator $E+A_nA^{-n}$ is reversible in space H.

Operator bundle $P(\lambda)$ satisfying conditions 1)-4) has discrete spectrum with unit limit point in infinity (see, for example, [1], [2], [3]).

Let us denote by $K(\Pi_-)$ the system of eigen and joined vectors responding to eigen values from the left half-plane

$$\Pi_-=\{\lambda\,|\,\mathrm{Re}\,\lambda<0\}.$$

Let $\lambda_q\in\Pi_-$ be eigen values of operator bundle (1), and $\{x_{q,0},\ldots,x_{q,m_q}\}$ be the corresponding system of eigen and associated vectors.

Definition. *We will name the system $K(\Pi_-)$ amplified k-multiple completed in H, if the system of vectors $\widetilde{x}_{q,s}=\{x_{q,s}^{(0)},x_{q,s}^{(1)},\ldots,x_{q,s}^{(k-1)}\}$, $s=0,1,\ldots,m_q$ constructed by the rule*

$$x_{q,s}^{(l)}=\frac{d^l}{dt^l}\left(e^{\lambda_q t}\sum_{h=0}^{s}\frac{t^{s-h}}{(s-h)!}x_{q,h}\right)\Bigg|_{t=0},\quad l=0,1,\ldots,k-1,\ s=0,1,\ldots,m_q$$

is completed in space $\widetilde{H}=\bigoplus_{m=0}^{k-1}H_{n-m-\frac{1}{2}}$.

Here spaces $H_\alpha=\{x:x\in D(A^\alpha),\ (x,y)_\alpha=(C^\alpha x,C^\alpha y)\}$, where C is the positive determined operator from polar expansion of operator $A: A=UC$.

For proof of the main result of the work we will reduce some lemmas which give an opportunity to estimate the resolvents of operator bundle $P(\lambda)$ and they are used for proof of the theorem on multiple completeness of system $K(\Pi_-)$.

Lemma 1. *On rays* $\Gamma_m=\left\{\lambda\,\middle|\,\arg\lambda=\frac{\pi}{2}+\frac{2\pi m}{n}\right\}$, $m=0,1,...k$ the estimation has place:

$$\left\|\lambda^j A^{n-j}P_0^{-1}(\lambda)\right\|\le b_j(\varepsilon),\ j=0,1,...,n-1, \tag{2}$$

where

$$b_0(\varepsilon)=\begin{cases}1 & \text{for}\quad 0\le\varepsilon<\dfrac{\pi}{2n},\\ \left(\sqrt{2}\cos k\varepsilon\right)^{-1} & \text{for}\quad \dfrac{\pi}{2n}\le\varepsilon<\dfrac{\pi}{n},\end{cases} \tag{3}$$

$$b_j(\varepsilon)=d_{n,j}(\cos k\varepsilon)^{-1},\ d_{n,j}=\left(\frac{j}{n}\right)^{\frac{j}{n}}\left(\frac{n-j}{n}\right)^{\frac{n-j}{n}},\ j=1,2,...,n-1. \tag{4}$$

Proof. Let $\lambda=re^{i\left(\frac{\pi}{2}+\frac{2\pi m}{n}\right)}$, then using spectral expansion of operator A

$$d_j(\lambda)=\left\|\lambda^j A^{n-j}P_0^{-1}(\lambda)\right\|=\left\|r^jA^{n-j}\left(r^nE+A^n\right)^{-1}\right\|=\sup_{\mu\in\sigma(A)}\left|r^j\mu^{n-j}\left(r^n+\mu^n\right)^{-1}\right|=$$

$$=\sup_{\xi>0,|\varphi|<\varepsilon}\left|r^j\xi^{n-j}\left(r^n+\xi^ne^{in\varphi}\right)^{-1}\right|=\sup_{\xi>0,|\varphi|<\varepsilon}\left|r^j\xi^{n-j}\left(r^{2n}+\xi^{2n}+2r^n\xi^n\cos n\varphi\right)^{-\frac{1}{2}}\right|\le$$

$$\le\sup_{\xi>0}\left|r^j\xi^{n-j}\left(r^{2n}+\xi^{2n}+2r^n\xi^n\cos n\varepsilon\right)^{-\frac{1}{2}}\right|.$$

For $j=0$ and $0\le\varepsilon<\frac{\pi}{2n}$, $\cos n\varepsilon>0$, and so

$$d_0(\lambda)\le\left|\xi^n\left(\xi^{2n}+r^{2n}\right)^{-1}\right|\le1,$$

and for $\frac{\pi}{2n}\le\varepsilon<\frac{\pi}{n}$, $\cos n\varepsilon\le0$ and so the estimation has place

$$d_0(\lambda)\le\left|\xi^n\left(\xi^n+r^{2n}+\left(\xi^n+r^{2n}\right)\cos n\varepsilon\right)^{-\frac{1}{2}}\right|=$$

$$=\left(1+\cos n\varepsilon\right)^{-\frac{1}{2}}\sup_{\xi>0}\left|\xi^n\left(\xi^{2n}+r^{2n}\right)^{-\frac{1}{2}}\right|\left(\sqrt{2}\cos k\varepsilon\right)^{-1}.$$

For $j=1,...,n-1$ we have

$$d_j(\lambda)\le\sup_{\xi>0}\left|\frac{\xi^jr^{n-j}}{\xi^n+r^n}\right|=\sup_{\xi>0}\left|\frac{\xi^n+r^{2n}+2\xi^nr^n}{\xi^n+r^{2n}+2\xi^nr^n\cos n\varepsilon}\right|^{\frac{1}{2}}=$$

$$=d_{n,j}\sup_{\xi>0}\left(1+\frac{2\xi^nr^n(1-\cos n\varepsilon)}{\xi^n+r^{2n}+2\xi^nr^n\cos n\varepsilon}\right)^{\frac{1}{2}}\le d_{n,j}\sup_{\xi>0}\left(1+\frac{2\xi^nr^n(1-\cos n\varepsilon)}{2\xi^nr^{2n}+2\xi^nr^n(1+\cos n\varepsilon)}\right)^{\frac{1}{2}}=$$

$$=d_{n,j}(\cos k\varepsilon)^{-\frac{1}{2}}.$$

Lemma 2. *Let conditions 1)-3) be fulfilled and the estimation has place*

$$\chi(\varepsilon)=\sum_{j=1}^{n}b_{n-j}(\varepsilon)\left\|B_j\right\|<1. \tag{5}$$

Then on sectors $\Gamma_{m,\theta} = \left\{ \lambda \left| \left| \arg\lambda - \left(\frac{\pi}{2} + \frac{2\pi m}{n} \right) \right| < \theta \right. \right\}$, $m = 0,1,\dots,k$ *for sufficient small* θ *the estimations have place*

$$\left\| P^{-1}(\lambda) \right\| \le const \left| \lambda \right|^{-n}, \tag{6}$$

$$\left\| A^n P^{-1}(\lambda) \right\| \le const\,. \tag{7}$$

Proof. By lemma 1 on rays $\Gamma_m = \left\{ \arg\lambda = \frac{\pi}{2} + \frac{2\pi m}{n} \right\}$, $m = 0,1,\dots,k$ the estimation (2) has place. On the other hand

$$P(\lambda) = P_0(\lambda) + P_1(\lambda) = \left(E + P_1(\lambda) \cdot P_0^{-1}(\lambda) \right) P_0(\lambda),$$

so by lemma 1 the estimation on these rays has place:

$$\left\| P_1(\lambda) \cdot P_0^{-1}(\lambda) \right\| = \left\| \sum_{j=1}^{n} \lambda^{n-j} A_j \left(P_0^{-1}(\lambda) \right)^{-1} \right\| \le \sum_{j=1}^{n} \left\| B_j \right\| \cdot \left\| \lambda^{n-j} A^j P_0^{-1}(\lambda) \right\| \le \sum_{j=1}^{n} b_{n-j} \left\| B_j \right\| = \chi(\varepsilon) < 1\,.$$

Therefore, bundle $P(\lambda)$ is reversible on these rays and

$$P^{-1}(\lambda) = P_0^{-1}(\lambda) \left(E + P_1(\lambda)\ P_0^{-1}(\lambda) \right) \tag{8}$$

and so

$$\left\| P^{-1}(\lambda) \right\| = \left\| P_0^{-1}(\lambda) \right\| \cdot \frac{1}{1 - \chi(\varepsilon)}.$$

On the other hand on rays Γ_m , $m = 0,1,\dots,k$ the estimation has place:

$$\left\| P_0^{-1}(\lambda) \right\| = \left\| \left(\xi^n E + A^n \right)^{-1} \right\| = \sup_{\xi > 0} \left| \left(r^{2n} + \xi^{2n} + 2^n r^n \xi^n \cos n\varepsilon \right)^{-\frac{1}{2}} \right| \le$$

$$\le \sup_{\xi > 0} \left| r^n \left(r^{2n} + \xi^{2n} + 2 r^n \xi^n \cos n\varepsilon \right)^{-\frac{1}{2}} \right| \cdot \left| r \right|^{-n} \le b_0(\varepsilon) \cdot \left| \lambda \right|^{-n},$$

where $b_0(\varepsilon)$ is determined from equality (3). Further for $\lambda \in \Gamma_m$

$$P\left(\lambda e^{i\theta} \right) = \left(E + \sum_{j=1}^{n-1} \lambda^{n-1} A_j P^{-1}(\lambda) \left(e^{i(n-j)\theta} - 1 \right) \right) P(\lambda) \tag{9}$$

and for $0 \le \theta < \frac{\pi}{2(n-1)}$ the estimation has place (see lemma 1):

$$\left\| \sum_{j=1}^{n-1} \lambda^{n-j} A_j \left({}^{i(n-j)} - 1 \right) P^{-1}(\lambda) \right\| = \left\| \sum_{j=1}^{n-1} \lambda^{n-j} B_j A^{n-j} P_0^{-1}(\lambda) \left(P(\lambda) P_0^{-1}(\lambda) \right)^{-1} \right\| \le$$

$$\le 2 \sin(k-1)\theta \sum_{j=1}^{n-1} \left\| B_j \right\| \cdot b_{n-j}(\varepsilon) \cdot \left\| \left(E + \sum_{j=1}^{n} \lambda^{n-j} A_j P^{-1}(\lambda) \right)^{-1} \right\| \le \frac{2\chi(\varepsilon)}{1 - \chi(\varepsilon)} \sin(k-1)\theta\,,$$

where $\chi(\varepsilon)$ is determined from inequality (5). So, choosing θ sufficient small (for fixed ε) so that $\frac{2\chi(\varepsilon)}{1-\chi(\varepsilon)} \sin(k-1)\theta < 1$.

Consequently for sufficient small $\theta \ge 0$ bundle $P\left(\lambda e^{i\theta} \right)$ is reversible and from (8) it follows that

$$\left\| P^{-1}\left(\lambda e^{i\theta} \right) \right\|^{-1} \le C(\varepsilon; \theta) \left| \lambda \right|^{-n}.$$

Inequality (7) is proved analogously.

Now we can prove the theorem on multiple completeness of system $K(\Pi_-)$.

Theorem 1. *Let conditions 1)-3) be fulfilled and the inequality has place*

$$\alpha(\varepsilon)=\sum_{j=1}^{n} C_{n-j}(\varepsilon)\|B_j\|<1, \tag{10}$$

where $C_0(\varepsilon)=\begin{cases} 1 & \text{for } 0\le\varepsilon<\dfrac{\pi}{2n}, \\ \left(\sqrt{2}\cos k\varepsilon\right) & \text{for } \dfrac{\pi}{2n}\le\varepsilon<\dfrac{\pi}{n}, \end{cases}$

for $j=1,2,...,k$

$$c_j(\varepsilon)=\begin{cases} (2\cos k\varepsilon)^{-\frac{j}{k}} & \text{for } 0\le\varepsilon<\dfrac{\pi}{2n}, \\ \left(2^{\frac{j-k}{n}}\cos k\varepsilon\right)^{-1} & \text{for } \dfrac{\pi}{2n}\le\varepsilon<\dfrac{\pi}{n}, \end{cases}$$

and for $j=k+1,...,n-1$

$$c_j(\varepsilon)=\begin{cases} 2^{\frac{(n-j)(k(j-k)-2)}{n}}(\cos k\varepsilon)^{\frac{j-n}{2}} & \text{for } 0\le\varepsilon<\dfrac{\pi}{2n}, \\ 2^{\frac{(n-j)(k(j-k)-1)}{n}}(\cos k\varepsilon)^{-1} & \text{for } \dfrac{\pi}{2n}\le\varepsilon<\dfrac{\pi}{n}. \end{cases}$$

If besides them one of the following conditions is fulfilled

1) $A^{-1}\in\sigma_p$ $(0<p\le k)$;

2) $B_j=A_jA^{-j}\in\sigma_\infty(H)$, and $A^{-1}\in\sigma_p$ $(0<p<\infty)$,

then system $K(\Pi_-)$ is k-multiple amplified completed in H.

Proof. It is obvious, that all coefficients $C_j(\varepsilon)$ from inequality (8) is not less than $b_j(\varepsilon)$ coefficients from inequality (6), i.e. for $0\le\varepsilon<\dfrac{\pi}{n}$ always $b_j(\varepsilon)\le c_j(\varepsilon)$, $j=0,1,...,n-1$. So for fulfillment of (10) all confirmations of Lemmas 1 and 2 have place. On the other hand it follows from fulfillment of (10) that the problem

$$P\left(\frac{d}{dt}\right)u(t)=0, \tag{11}$$

$$u^{(\nu)}=\varphi_\nu, \quad \nu=0,1,...,k-1 \tag{12}$$

has only regular solution from space $W_2^n(R_+;H)$ (see [1]), for any $\varphi_\nu\in D\left(C^{n-\nu-\frac{1}{2}}\right)$, $\nu=0,1,...,k-1$ (see [8]). Let us denote this solution by $u(t)$.

From lemma 1 it follows that it is possible to represent it in the form

$$u(t)=\frac{1}{2\pi i}\int_{\Gamma_{\pm\left(\frac{\pi}{2}+\theta\right)}}\hat{u}(\lambda)e^{\lambda t}d\lambda,$$

where $\Gamma_{\pm\left(\frac{\pi}{2}+\theta\right)}=\left\{\lambda:\arg\lambda=\pm\left(\frac{\pi}{2}+\theta\right)\right\}$, and θ is the chosen sufficient small number and

$$\hat{u}(\lambda)=P^{-1}(\lambda)\sum_{p=0}^{n-1}Q_p(\lambda)U^{(p)}(0),\ Q_p(\lambda)=\lambda^{-(p+1)}\left(P(\lambda)-\sum_{s=0}^{p}\lambda^s A_{n-s}\right),\ p=0,1,...,n-1.$$

Let there exists vector $\tilde{\varphi}\in\tilde{H}=\overset{k-1}{\underset{j=0}{\oplus}}H_{n-j-\frac{1}{2}}$ orthogonal to system $K(\Pi_-)$ in space $\tilde{H}$. Then it is easy to see that vector-function

$$R(\lambda)=\sum_{j=0}^{n-1}\left(C^{n-j-\frac{1}{2}}P^{-1}(\bar{\lambda})\right)^*\lambda^j C^{n-j-\frac{1}{2}}\varphi_j\,,\ \left(\tilde{\varphi}=(\varphi_1,\varphi_2,...,\varphi_{k-1})\right)$$

is holomorphic in half-plane Π_- (see [1]). Then for the scalar function

$$\varphi(t)=\sum_{j=0}^{k-1}\left(C^{n-j-\frac{1}{2}}u^{(j)}(t),C^{n-j-\frac{1}{2}}\varphi_j\right)=\frac{1}{2\pi i}\int_{\Gamma_{\pm\left(\frac{\pi}{2}+\theta\right)}}\left(\sum_{p=0}^{n-1}Q_p(\lambda)u^{(p)}(0),\,R(\bar{\lambda})\right)e^{\lambda t}d\lambda$$

we can confirm that it is equal to zero for $t>0$. Really, as far as function $\left(\sum_{p=0}^{n-1}Q_p(\lambda)u^{(p)}(0),\,R(\bar{\lambda})\right)=q(\lambda)$ is holomorphic in left half-plane Π_- and it is holomorphic in right half-plane as Laplace transformation of function from $L_2(R_+;H)$, i.e. $q(\lambda)$ is an entire function. From the condition of theorem it follows that this function is entire in order $p'\le p<\infty$ and in each of the angles $\arg\lambda=\frac{\pi}{2}+\frac{2\pi m}{n}$, $m=0,1,...,k-1$ and $\Gamma_{\pm\left(\frac{\pi}{2}+\theta\right)}$ does not grow faster than $|\lambda|^n$ and consequently by Fragman-Lindelef's theorem (see [9]) $q(\lambda)$ is the polynomial with power no more than n. Then hence we conclude that $\varphi(t)=0$ for $t>0$, as far as

$$\int_{\Gamma_{\pm\left(\frac{\pi}{2}+\theta\right)}}q(\lambda)e^{\lambda t}d\lambda=\int_{\Gamma_{\pm\left(\frac{\pi}{2}+\theta\right)}}\sum_{j=0}^{n}\lambda^j a_j e^{\lambda t}dt=0.$$

Passing to limit for $t\to 0$ we obtain that

$$\varphi(0)=\sum_{j=0}^{k-1}\left(C^{n-j-\frac{1}{2}}u^{(j)}(0),C^{n-j-\frac{1}{2}}\varphi_j\right)=\sum_{j=0}^{k-1}\left(\varphi_j,\varphi_j\right)_{H_{n-j-\frac{1}{2}}}=(\varphi,\varphi)_{\tilde{H}}=0,\quad \text{i.e.}\quad \varphi=0.$$

The theorem has been proved.

Let us remark that for $\varepsilon=0$ our result coincides with the results of [5] and amplifies the results of [6] for $n=4$ and [7] for $n=2$.

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