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ON THE ASYMPTOTICS OF SOLUTION OF A SYSTEM OF VOLTERRA TYPE SINGULAR PERTURBED INTEGRO-DIFFERENTIAL EQUATIONS

Abstract

In the given paper constructed asymptotics of the solution of a system of singular-perturbed integro-differential equations of Volterra type and shown that integral terms play main role for boundedness of solution as $\mu \to 0$.

1. Statement of the problem. It is known that for the stability of any solution of the problem

$$\frac{dx}{dt} = Ax + B(t), \qquad x(0) = x_0,\ 0 \le t \le T \tag{1^0}$$

it is necessary and sufficient that all real parts of all characteristic numbers of the matrix A were negative. This condition plays an important role in studying asymptotic behavior of the solution of a singularly perturbed system

$$\mu\frac{dz}{dt} = A(t)z + B(t), \qquad z(0,\mu) = z_0,\ 0 \le t \le T, \tag{2^0}$$

where $\mu > 0$ is a small parameter.

By fulfilling the stability condition the asymptotic behavior of the solution of an initial value problem for singularly perturbed integro-differential equations is completely analogous to the asymptotic behavior of the solution of problem (2^0) [see 1-3]. The asymptotic expansion of solutions of these two problems is of the same form, and the difference is only in complication of construction algorithm for the asymptotic expansion in integro-differential equation case.

In this paper, we also study asymptotics of solution of a singularly perturbed Volterra type integro-differential equation, but of the other type initial conditions.

Consider the next problem

$$\mu\frac{dz}{dt} = A(t)z + \int_0^t k(t,s)z(s,\mu)ds + B(t), \tag{1}$$

$$z(T,\mu) = z_0, \tag{2}$$

where $0 \le t \le T$, $0 \le s \le T$, $A(t)$ and $k(t,s)$ – $n \times n$-dimensional matrices, $B(t)$ is n-dimensional vector and they are sufficiently smooth, $0 < \mu$ is a small parameter. Besides, assume that $k(t,s) \ne 0$, and characteristic values $\lambda_i(t)$ of the matrix $A(t)$ satisfy the conditions:

$$\operatorname{Re}\lambda_i(t) < 0,\ i = \overline{1,n},\ t \in [0,T]. \tag{3}$$

It is clear that for $k(t,s) = 0$ (i.e. in the absence of integral members) the solution of problem (1), (2), under condition (3), generally speaking, is unrestricted for $\mu \to 0$. In the suggested paper it is shown that problems (1), (2) under condition (3) have solution restricted for $\mu \to 0$, and a limit function is the solution of the following integral equations system

$$O = A(t)z(t) + \int_0^t k(t,s)z(s)ds + B(t) + A^{-1}(0)k(t,0)a,$$

where a is defined by the data of the problem. With that end in view first consider an auxiliary problem.

2. Statement and solution of the auxiliary problem. At first consider equation system (1) with the following initial value of the solution:

$$z(0,\mu) = \frac{1}{\mu}a(\mu), \tag{4}$$

where $a(\mu)$ is representable in the form of asymptotic series in powers of μ for $\mu \to 0$

$$a(\mu) = a_{-1} + \mu a_0 + \ldots + \mu^{k+1}a_k + \ldots. \tag{5}$$

In the given case under $a(\mu)$ we understand n-dimensional vector-column, whose each component $a^{(i)}(\mu),\ i=\overline{1,n}$ is represented in the form of asymptotic series in powers of μ for $\mu \to 0$

$$a^{(i)}(\mu) = a_{-1}^{(i)} + \mu a_0^{(i)} + \ldots + \mu^{k-1}a_k^{(i)} + \ldots \qquad i = 1,n, \tag{5.1}$$

where $a_k^{(i)}$ are the components of $a_k - n$-dimensional vector-column $k=-1,0,1,\ldots,\ i=1,n$.

Then the components of the vector-column $z(0,\mu)$ are

$$z_i(0,\mu) = \frac{1}{\mu}a^{(i)}(\mu),\ i=1,n. \tag{6}$$

We shall seek the solution of problem (1), (3), (4), in the form of the following series

$$\begin{aligned} z(t,\mu) = \bar z_0(t) + \mu\bar z_1(t) + \ldots + \mu^k \bar z_k(t) + \ldots + \frac{1}{\mu}\Pi_{-1}z(\tau) + \\ + \Pi_0 z(\tau) + \mu\Pi_1 z(\tau) + \ldots + \mu^k \Pi_k z(\tau) + \ldots, \end{aligned} \tag{7}$$

where $\tau = \frac{t}{\mu}$.

Similar to abovestated ones, here the components of vector $z(t,\mu)$ are $z_i(t,\mu),\ i=1,n$, each of them is represented as

$$\begin{aligned} z_i(t,\mu) = \bar z_0^{(i)}(t) + \mu\bar z_1^{(i)}(t) + \ldots + \mu^k \bar z_k^{(i)}(t) + \ldots + \frac{1}{\mu}\Pi_{-1}^{(i)}z(\tau) + \\ + \Pi_0^{(i)}z(\tau) + \mu\Pi_1^{(i)}z(\tau) + \ldots + \mu^k\Pi_k^{(i)}z(\tau) + \ldots \qquad i=\overline{1,n}. \end{aligned} \tag{8}$$

Here $z_k^{(i)}(t)$ and $\Pi_k^{(i)}z(\tau)$ are the components of vectors $z_k(t)$ and $\Pi_k z(\tau),\ i=1,n$ respectively.

Now by substituting (7) into (1) and taking into account all aboveadopted denotations we get the following

$$\mu\frac{d}{dt}\left[\bar z_0(t) + \mu\bar z_1(t) + \ldots + \mu^k\bar z_k(t) + \ldots\right] +$$

$$+\frac{d}{d\tau}\left[\frac{1}{\mu}\Pi_{-1}z(\tau) + \Pi_0 z(\tau) + \ldots + \mu^k\Pi_k z(\tau) + \ldots\right] =$$

$$= A(t)\left[\bar z_0(t) + \ldots + \mu^k\bar z_k(t) + \ldots\right] + A(\tau\mu)\left[\frac{1}{\mu}\Pi_{-1}z(\tau) + \Pi_0 z(\tau) + \ldots + \mu^k\Pi_k z(\tau) + \ldots\right] +$$

$$+\int_0^t k(t,s)\left[\bar{z}_0(s)+\ldots+\mu^k\bar{z}_k(s)+\ldots\right]ds+\mu\int_0^\infty k(t,\sigma\mu)\left[\frac{1}{\mu}\Pi_{-1}z(\sigma)+\ldots+\mu^k\Pi_k z(\sigma)+\ldots\right]d\sigma-$$

$$-\mu\int_\tau^\infty k(\tau\mu,\sigma\mu)\left[\frac{1}{\mu}\Pi_{-1}z(\sigma)+\Pi_0 z(\sigma)+\ldots+\mu^k\Pi_k z(\sigma)+\ldots\right]d\sigma+B(t), \qquad (9)$$

where $\sigma=s/\mu$.

By representing the right hand side of (9) in the form of series of powers of μ and equating coefficients under the same powers of μ in both hand sides of equality (9) (moreover, separately depending on τ, and separately depending on t), we get an equation to determine $\Pi_{k-1}z(\tau)$, $\bar{z}_k(t)$ $(k=0,1,\ldots)$. In an analogous manner, by substituting series (7) into (4), replacing $a(\mu)$ by expression (5), taking into account all denotations and equating coefficients under the same powers of μ, we get an initial condition for the terms of series (7).

For $\Pi_{-1}z(\tau)$ we get a system of differential equations

$$\frac{d\Pi_{-1}z}{d\tau}=A(0)\Pi_{-1}z$$

with initial conditions $\Pi_{-1}z(0)=a_{-1}$, whence

$$\Pi_{-1}z(\tau)=\exp(A(0)\tau)a_{-1} \quad (\tau\geq 0). \qquad (10)$$

Since $A(0)$ is a constant $n\times n$-dimensional matrix, then $\exp(A(0)\tau)$ is a matrix exponent.

For $\bar{z}_0(t)$ we get an integral equation

$$O=A(t)\bar{z}_0(t)+\int_0^t k(t,s)\bar{z}_0(s)ds+\int_0^\infty k(t,0)\Pi_{-1}z(\sigma)d\sigma+B(t). \qquad (11)$$

Now substitute here expression (10) for $\Pi_{-1}z(\tau)$ and write equation in the form

$$\bar{z}_0(t)=\int_0^t \bar{k}(t,s)\bar{z}_0(s)ds+f_0(t), \qquad (12)$$

where $\bar{k}(t,s)=-A^{-1}(t)k(t,s)$, $f_0(t)=\left[-A^{-1}(0)k(t,0)\right]a_{-1}-A^{-1}(t)B(t)$.

By $R(t,s)$ denote a resolvent of the Kernel $\bar{k}(t,s)$.

The solutions of equations (12) write in the form

$$\bar{z}_0(t)=f_0(t)+\int_0^t R(t,s)f_0(s)ds.$$

Substituting here an expression for $f_0(t)$, get

$$\bar{z}_0(t)=-A^{-1}(0)\left[\overline{K}(t,0)+\int_0^t\overline{R}(t,s)\overline{K}(s,0)ds\right]a_{-1}+$$

$$+\left[-A^{-1}(t)B(t)-\int_0^t A^{-1}(s)B(s)\overline{R}(t,s)ds\right].$$

Since for $R(t,s)$ it is valid the following

$$R(t,s)=K(t,s)+\int_s^t R(t,p)K(p,s)dp,$$

then coefficient for a_{-1} equals to $-A^{-1}(0)R(t,0)$. Then

$$\bar{z}_0(t)=-A^{-1}(0)\bar{R}(t,0)a_{-1}+\widetilde{Z}_0(t), \tag{13}$$

where

$$\widetilde{Z}_0(t)=-A^{-1}(t)B(t)-\int_0^t A^{-1}(s)B(s)\bar{R}(t,s)ds.$$

As we see from (13), $\bar{z}_0(t)$ linearly depends on a_{-1}, this linear dependence further will be used in consideration of the main problem (1), (2), (3).

Analogously we can determine $\Pi_0 z(\tau), \bar{z}_1(t), \Pi_1 z(\tau), \bar{z}_2(t), \ldots$. To determine $\Pi_k z(\tau)$ at each $(k=0,1,2,..)$ we get differential equation

$$\frac{d\Pi_k Z}{d\tau}=A(0)\Pi_k Z+P_k(\tau)\exp[A(0)\tau],$$

where $P_k(\tau)$ is some known n-dimensional vector-column with initial conditions $\Pi_k z(0)=a_k-\bar{z}_k(0)$, hence we get

$$\Pi_k z(\tau)=\exp(A(0)\tau)[a_k-z_k(0)]+\exp(A(0)\tau)\left[\int_0^\tau P_k(\sigma)d\sigma\right]. \tag{14}$$

To determine the functions $z_k(t)$ at each $(k=1,2,\ldots)$ we get integral equations analogous to (12) for $z_0(t)$

$$z_k(t)=\int_0^t K(t,s)z_k(s)ds+f_k(t), \tag{15}$$

where $f_k(t)$ is expressed by the problem datas in certain manner.

Acting like in definition of $z_0(t)$, we get

$$\bar{z}_k(t)=-A^{-1}(0)\bar{R}(t,0)a_{k-1}+\widetilde{Z}_k(t), \tag{16}$$

where $\widetilde{Z}_k(t)$ is a vector-function.

The linear dependence of $z_k(t)$ on a_{k-1} will be also used below in considering the main problem (1), (2), (3).

Assuming $A(t), B(t), k(t,s)$ sufficiently smooth, determine the terms of series (7) up to number n inclusively and denote by $Z_n(t,\mu)$ a partial sum of order n of series (7):

$$Z_n(t,\mu)=\frac{1}{\mu}\Pi_{-1}z(\tau)+\sum_{k=0}^n \mu^k\left[z_k(t)+\Pi_k z(\tau)\right]. \tag{17}$$

Theorem 1. *There will be found such constants $\mu_0>0$ and $c>0$, that for $0<\mu\le\mu_0$ the solution $z(t,\mu)$ of auxiliary problem (1), (3), (4) uniquely exists and it satisfies the inequality*

$$\|z(t,\mu)-Z_n(t,m)\|\le c\mu^{n+1} \quad for \quad 0\le t\le T. \tag{18}$$

Proof. Since equations (1) are linear, then it is clear that the solution exists and it is unique.

Let's prove inequality (18). Represent the following denotations $u(t,\mu)=z(t,\mu)-Z_n(t,\mu)$. Substituting into (1) $z=u+Z_n$ we get the following equation system with respect to $u(t,\mu)$

$$\mu \frac{du}{dt} = A(t)u + \int_0^t K(t,s)u(s,\mu)ds + H(t,\mu), \tag{19}$$

where

$$H(t,\mu) = A(t)Z_n(t,\mu) + \int_0^t K(t,s)Z_n(s,\mu)ds + B(t) - \mu \frac{dZ_n}{dt}.$$

Here $u(t,\mu), Z_n(t,\mu)$ and $H(t,\mu)$ are n-dimensional vector-functions. Substituting into $H(t,\mu)$ expression (17) for $Z_n(t,\mu)$ using the equation for $\Pi_k z(\tau)$ $(k=-1,0,...,n)$ and $z_k(t)$ $(k=0,1,...,n)$ and making replacement $t=\tau\mu$, $s=\sigma\mu$ in corresponding summands we get the following estimate

$$\|H(t,\mu)\| \le c\mu^{n+1} \quad \text{for} \quad 0 \le t \le T,\ 0<\mu\le\mu_0, \tag{20}$$

$$u(0,\mu) = \left(\mu^{n+2}a_{n+1} + ...\right)\frac{1}{\mu},$$

whence

$$\|u(0,\mu)\| \le c\mu^{n+1} \quad \text{for} \quad 0<\mu\le\mu_0. \tag{21}$$

The solution of system (19) we may write in the form of

$$u(t,\mu) = \Phi(t,0,\mu)u(0,\mu) + \int_0^t \frac{1}{\mu}\Phi(t,s,\mu)H(s,\mu)ds. \tag{22}$$

where $\Phi(t,s,\mu)$ is the solution of the homogeneous system

$$\mu \frac{d\Phi(t,s,\mu)}{dt} = A(t)\Phi(t,s,\mu) + \int_s^t K(t,p)\Phi(p,s,\mu)dp \quad (0\le s\le t\le T)$$

satisfying the condition $\Phi(s,s,\mu)=E_n$, where E_n is a unique matrix.

It is known that $\Phi(t,s,\mu)$ satisfies the inequality

$$\|\Phi(t,s,\mu)\| \le c\left[\mu + \exp\left(\frac{-\chi(t-s)}{\mu}\right)\right], \tag{23}$$

$$0\le s\le t\le T,\ 0<\mu\le\mu_0,\ \chi>0.$$

Form (22) by virtue of inequalities (20), (21), (23) it immediately follows

$$\|u(t,\mu)\| \le c\mu^{n+1} \quad \text{for} \quad 0\le t\le T,\ 0<\mu\le\mu_0,$$

that proves (18) and thereby same theorem 1.

3. The solution of the main problem. Now return to the main problem (1), (2), (3). For their solution we use already constructed asymptotics of the solution of auxiliary problem (1), (3), (4). Denote this solution by $z(t,a,\mu)$. We must select such a vector a that $z(t,a,\mu)$ satisfies condition (2). Thus, with respect to a we get the following system

$$z(T,a,\mu) = z_0. \tag{24}$$

We shall seek solution (24) in the form of the following vector

$$a = a_{-1} + \mu a_0 + ... + \mu^{n+1}a_k + ... \ .$$

To find vector-coefficient $a_{-1}, a_0, ...$ substitute into (24) instead of precise solution $z(T,a,\mu)$, its asymptotic expansion (7). Since at the point $t=T$ all boundary vector-functions have the estimate

$$\|\Pi_k z(T/\mu)\| \le c\exp(-\chi T/\mu),\ 0<\chi\le\|A(0)\|,$$

then (24) has the form

$$z_0(T) + \mu z_1(T) + ... + \mu^k z_k(T) + ... = z_0.$$

Now taking into account the dependence $z_k(T)$ on a_{k-1} [(13) and (16)] with respect to $a_{-1}, a_0, \ldots$ we get a system of linear equations

$$z_0(T) = -A^{-1}(0)R(T,0)a_{-1} + \tilde{z}_0(T) = z_0, \quad (25)$$

$$z_k(T) = -A^{-1}(0)R(T,0)a_{k-1} + \tilde{z}_k(T) = 0 \quad (k = 1,2,\ldots). \quad (26)$$

Let $R^{-1}(T,0)$ exist. Then equation (25), (26) are uniquely solvable with respect to a_{-1}, a_{k-1}. Denote these solutions $\bar{a}_k$, $k = -1,0,1,\ldots$, (each of them is a vector column) and construct $Z_n(t,\mu)$ determined by the formula (17), for $a_{-1} = \bar{a}_{-1}$, $a_0 = \bar{a}_0, \ldots, a_n = \bar{a}_n$.

Now consider the solution $z(t, \bar{a}_{-1}, \mu)$ of system (1) with the initial condition

$$z(0, \bar{a}_{-1}, \mu) = \frac{1}{\mu} \bar{a}_{-1} \ (\bar{a}_{-1} \text{ is } n \text{ dimensional vector column}).$$

For it at $t = T$ we have

$$z(T, \bar{a}_{-1}, \mu) = z_0(T) + o(\mu) = z_0 + o(\mu),$$

since $\bar{a}_{-1}$ was chosen so that $z_0(T) = z_0$. Hence, the solution $z(t, \bar{a}_{-1}, \mu)$ satisfies the condition (2) to within $o(\mu)$. Thus, there exists a unique value of $a = a(\mu) = \bar{a}_{-1} + o(\mu)$ such that

$$z(T, a(\mu), \mu) = z_0.$$

This means that the solution $Z(t,\mu)$ of system (1) with an initial condition $Z(0,\mu) = \frac{1}{\mu}\bar{a}(\mu)$ satisfies the condition (2), i.e.

$$Z(T,\mu) = z_0.$$

Thereby we proved the existence and uniqueness of the solution of problem (1), (2), (3).

Now consider the solution $z(t, a, \mu)$ for

$$a = (a)_n \equiv a_{-1} + \mu a_0 + \ldots + \mu^{n+1} a_n.$$

For it at $t = T$ by virtue of equations (25), (26), where $k = 1,2,\ldots,n+1$ we have

$$z(T,(a)_n,\mu) = z_0(T) + \mu z_1(T) + \ldots + \mu^{n+1} z_{n+1}(T) + o(\mu^{n+2}) = z_0 + o(\mu^{n+2}).$$

Hence it follows the inequality

$$\|a(\mu) - (a)_n\| \le c\mu^{n+2}, \quad \text{i.e. for } a(\mu)$$

it is valid the asymptotic representation

$$\bar{a}(\mu) = \bar{a}_{-1} + \mu\bar{a}_0 + \ldots + \mu^{n+1} a_n + o(\mu^{n+2}).$$

From theorem 1 now it follows the following

Theorem 2. *If there exists* $R^{-1}(T,0)$ *then we can find such constants* $\mu_0 > 0$, $c > 0$, *that for* $0 < \mu \le \mu_0$ *there exists a unique solution* $Z(t,\mu)$ *of problem (1), (2), (3) and there holds the inequality*

$$\|Z(t,\mu) - Z_n(t,\mu)\| \le c\mu^{n+1} \ \textit{for} \ 0 \le t \le T. \quad (27)$$

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