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MULTIPLE COMPLETENESS OF EIGEN AND ADJOINT VECTORS SYSTEM OF SOME CLASSES OF POLYNOMIAL PENCILS

Abstract

In the Hilbert space the polynomial pencil is considered. This pencil is a derivative of the Keldysh's polynomial pencil. Under the definite conditions on the operators we prove the n-fold completeness of eigen and associated elements of this pencil.

This result is applied to the differential equations.

The completeness of eigen and adjoint (e.a.) vectors of a linear operator acting on a Hilbert space H is one of important directions of the spectral theory of linear operators.

And the multiple completeness of a system of e.a. vectors of polynomial pencils has a direct relation with solution of Cauchy's problem for operator-differential equations.

M.V. Keldysh [1] had a great contribution in this direction. He considered a polynomial pencil being the perturbation of a self-adjoint operator $\lambda^n B^n$ with a polynomial pencil of less order in spectral parameter λ.

Later, M.V. Keldysh's result was generalized in different directions.

More general result in a Hilbert space H is considered in this paper

$$L(\lambda)=\sum_{i=0}^{n-2}\lambda^i A_i B^i+\lambda^{n-1}(A_{n-1}-E)B^{n-1}+\lambda^n K B^{n-1}, \tag{1}$$

where A_i, B, K are linear operators in H.

Let $\{x_k\}_{k=1}^{\infty}$ be a system of e.a. vectors of the pencil (2.7). Starting with this, construct $n-1$ derivatives of the system $\{x_{r,k}\}_{k=1}^{\infty}$ by the following way: if x_0 is an eigen-vector, then

$$x_{r,0}=\alpha_r\left(1+\lambda e^{i\omega_1}\right)\ldots\left(1+\lambda e^{i\omega_r}\right)x_0, \tag{2}$$
$$\left(r=\overline{1,n-1},\ \omega_i=\frac{2\pi}{n-1}i\ \left(i=\overline{1,n-1}\right)\right);$$

if x_k is the k-th adjoint to the eigen vector x_0 of the pencil (2.7), then the vectors

$$\begin{aligned}x_{r,k}=\alpha_r\Bigg[&\left(1+\lambda e^{i\omega_1}\right)\cdots\left(1+\lambda e^{i\omega_r}\right)x_k+\frac{1}{1!}\frac{d}{d\lambda}\left(1+\lambda e^{i\omega_1}\right)\cdots\left(1+\lambda e^{i\omega_r}\right)x_{k-1}+\\&+\frac{1}{k!}\frac{d^k}{d\lambda^k}\left(1+\lambda e^{i\omega_1}\right)\cdots\left(1+\lambda e^{i\omega_r}\right)x_0\Bigg],\qquad r=1,2,\ldots,n-1\end{aligned} \tag{3}$$

correspond to the adjoint vector x_k in the r-th derivative of the system.

Thus, by system $\{x_i\}$, $n-1$ derivatives of the system $\{x_{r,k}\}_{k=1}^{\infty}$, $r=1,2,\ldots,n-1$ are determined. k-fold completeness of the system of e.a. vectors of the pencil (1) means there the completeness of the system $\{(x_k,x_{1,k},\ldots,x_{n-1,k})\}_{k=1}^{\infty}$ in a direct sum of n-copies of the space H.

Theorem 1. *Let be fulfilled the following conditions:*

a) *the operators A_i are completely continuous, the operators K and B are completely self-adjoint, having finite orders ρ_1 and ρ_2 respectively;*

b) *choose*

$$\sum_{r=0}^{n-1}\left|\frac{\alpha_r}{\alpha_{r+1}}\right| < \sin\varepsilon, \qquad \text{where} \qquad \varepsilon = \frac{\pi(\rho_2 + \rho_1(n-1))}{\rho_1\rho_2}, \ \alpha_0 = 1.$$

Then a system of e.a. vectors of the pencil (2.7) is n-fold complete in the space H.

Proof of Theorem 1. Denote by $\omega_1, \omega_2, ..., \omega_{n-1}$ the roots of $(n-1)$-th degree from the unit and by D_i denote the operators:

$$D_{n-1} = A_{n-1},$$

$$D_{n-2} = \left(A_{n-2} - A_{n-1} \sum_{1 \le k_i \le n-1} e^{i(\omega_{k_1} + \cdots + \omega_{k_{n-2}})} \cdot \frac{1}{\sum\limits_{1 \le k_i \le n-2} e^{i(\omega_{k_1} + \cdots + \omega_{k_{n-2}})}} \right), \tag{4}$$

....

$$D_0 + D_1 + \cdots + D_{n-1} = A_0 \ .$$

Later in the expression $e^{i(\omega_{k_1} + \cdots + \omega_{k_i})}$ from (4) and in similar expressions we shall assume all ω_{k_i} to be different.

Consequently, we can construct the operators $D_{n-k}, ..., D_1$ using before obtained operators $D_{n-1}, D_{n-2}, ..., D_{n-k+1}$ and ets. Note that the operators are chosen so that they are the solutions of the system

$$\begin{cases} D_0 + D_1 + \cdots D_{n-1} = A_0 \\ D_1 e^{i\omega_1} + D_2\left(e^{i\omega_1} + e^{i\omega_2}\right) + D_3\left(e^{i\omega_1} + e^{i\omega_2} + e^{i\omega_3}\right) + \\ + \cdots + D_{n-1}\left(e^{i\omega_1} + e^{i\omega_2} + \cdots + e^{i\omega_{n-1}}\right) = A_1 \\ \sum\limits_{k=m}^{n-1} D_k \left(\sum\limits_{1}^{k} e^{i(\omega_{n_1} + \cdots + \omega_{n_m})} \right) = A_m \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ D_{n-2} \sum\limits_{1}^{n-2} e^{i(\omega_{m_1} + \cdots + \omega_{m_{n-2}})} + D_{n-1} \sum\limits_{1}^{n-1} e^{i(\omega_{m_1} + \cdots + \omega_{m_{n-2}})} = A_{n-2} \\ D_{n-1} = A_{n-1} \end{cases} \tag{5}$$

in $e^{i(\omega_{m_1} + \cdots + \omega_{m_n})}$ the degree exponent is the sum of different ω_i multiplied by i. The operators $D_1, ..., D_{n-1}$ are completely continuous, since they are linear combinations of completely continuous operators.

In the space $\widetilde{H}$ being a direct sum of Hilbert spaces H consider the equation

$$\left(\widetilde{D} - \lambda\widetilde{B}\right)\widetilde{x} = \widetilde{x}, \tag{6}$$

where $\widetilde{x} = (x_0, x_1, ..., x_{n-1})$ and the operators $\widetilde{D}$ and $\widetilde{B}$ are given by means of operator matrices

$$\widetilde{D} \sim \begin{pmatrix} 0 & c_1 E & 0 & 0 & \dots & 0 \\ 0 & 0 & c_2 E & 0 & \dots & 0 \\ 0 & 0 & 0 & c_3 E & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & c_{n-1} E \\ \frac{D_0}{c_1 \dots c_{n-1}} & \frac{D_1}{c_2 \dots c_{n-1}} & \frac{D_2}{c_3 \dots c_{n-1}} & \frac{D_3}{c_4 \dots c_{n-1}} & \dots & D_{n-1} \end{pmatrix}. \tag{7}$$

In the expression (7) $c_i = \dfrac{\alpha_{i-1}}{\alpha_i}$, $i = 1,2,\dots,n-1$.

$$\widetilde{B} \sim \begin{pmatrix} e^{i\omega_1} B & 0 & \dots & 0 & 0 \\ 0 & e^{i\omega_2} B & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{i\omega_{n-1}} B & 0 \\ 0 & 0 & \dots & 0 & K \end{pmatrix}. \tag{8}$$

By virtue of self-adjointness of operators B and K the operator $\widetilde{B}$ is normal and its eigen-values Lie on the rays coming out from the origin of coordinates and passing through the roots of the $(n-1)$-th degree from (1).

Thus, we have $D_0, D_1, \dots, D_{n-1}$ that are completely continuous operators, c_i may be chosen sufficiently small, and $c_i = \dfrac{\alpha_{i-1}}{\alpha_i} \neq 0$, $(i = 1,2,\dots,n)$.

In the case when c_i are sufficiently small in modules, $\widetilde{D}$ is a completely continuous operator whose bounded part may be arbitrary small in the norm at the expense of the choice of numbers c_i.

Consequently the completeness of e.a. vectors of the equation (6) in the space $\widetilde{H}$ follows from [2].

Let $(x_0, \dots, x_{n-1})$ be an eigen-element (6), then

$$\begin{aligned} x_1 &= \frac{1}{c_1}\left(E + \lambda e^{i\omega_1} B\right) x_0 ; \\ x_2 &= \frac{1}{c_1 c_2}\left(E + \lambda e^{i\omega_1} B\right)\left(E + \lambda e^{i\omega_2} B\right) x_0 ; \\ &\dots\dots\dots\dots\dots\dots \\ x_{n-1} &= \frac{1}{c_1 c_2 \dots c_{n-1}}\left(E + \lambda e^{i\omega_1} B\right) \cdots \left(E + \lambda e^{i\omega_{n-1}} B\right) x_0 . \end{aligned} \tag{9}$$

Then

$$\frac{D_0}{c_1 c_2 \dots c_{n-1}} x_0 + \frac{D_1}{c_2 c_3 \dots c_{n-1}} x_1 + \frac{D_2}{c_3 c_4 \dots c_{n-1}} x_2 + \dots + D_{n-1} x_{n-1} = (E + \lambda K) x_{n-1}. \tag{10}$$

Substituting the values $x_1, x_2, \dots, x_{n-1}$ we have from (9) in (10)

$$\frac{D_0}{c_1 \dots c_{n-1}} x_0 + \frac{D_1\left(E + \lambda e^{i\omega_1} B\right)}{c_1 \dots c_{n-1}} x_0 + \frac{D_2\left(E + \lambda e^{i\omega_1} B\right)\left(E + \lambda e^{i\omega_2} B\right)}{c_1 \dots c_{n-1}} x_0 + \dots +$$

$$+\frac{D\left(E+\lambda e^{i\omega_1}B\right)\cdots\left(E+\lambda e^{i\omega_{n-1}}B\right)}{c_1\ldots c_n}x_0+\cdots+\frac{D_{n-1}\left(E+\lambda e^{i\omega_1}B\right)\cdots\left(E+\lambda e^{i\omega_{n-1}}B\right)}{c_1\ldots c_n}x_0=$$

$$=\frac{\left(E+\lambda K\right)\left(E+\lambda e^{i\omega_1}B\right)\cdots\left(E+\lambda e^{i\omega_{n-1}}B\right)}{c_1\ldots c_n}x_0 .$$

Opening the brackets and summing the coefficients at the same degrees λ, and taking into account the equality system (5), we get:

$$\left(A_0+\lambda A_1B+\lambda^2A_2B^2+\cdots+\lambda^{n-2}A_{n-2}B^{n-2}+\lambda^{n-1}\left(A_{n-1}-E\right)B^{n-1}-\lambda KB^{n-1}\right)x_0=x_0 .$$

In an analogous way, we can say that all adjoined elements of (6) to the eigen-element $\widetilde{x}$ are such that their first coordinates are the corresponding adjoined elements to the eigen element x_0 of the equation (11).

We have

$$\widetilde{y}=\widetilde{D}\widetilde{y}-\lambda\widetilde{B}\widetilde{y}-\widetilde{B}\widetilde{x},$$

$$y_0=c_1y_1-\lambda e^{i\omega_1}By_0-e^{i\omega_1}Bx_0,$$

$$y_1=\frac{1}{c_1}y_0+\frac{1}{c_1}\lambda e^{i\omega_1}By_0+\frac{1}{c_1}e^{i\omega_1}Bx_0=\frac{1}{c_1}\left(E+\lambda e^{i\omega_1}B\right)y_0+\frac{1}{c_1}e^{i\omega_1}Bx_0,$$

$$y_2=\frac{1}{c_2}\left(E+\lambda e^{i\omega_2}B\right)y_1+\frac{1}{c_2}e^{i\omega_2}Bx_1=\frac{1}{c_1c_2}\left(E+\lambda e^{i\omega_1}B\right)\left(E+\lambda e^{i\omega_2}B\right)y_0+$$

$$+\frac{1}{c_1c_2}e^{i\omega_1}b\left(E+\lambda e^{i\omega_2}B\right)x_0+\frac{1}{c_1c_2}e^{i\omega_2}B\left(E+\lambda e^{i\omega_1}B\right)x_0=\frac{1}{c_1c_2}\frac{d}{d\lambda}\left(E+\lambda e^{i\omega_1}B\right)\times$$

$$\times\left(E+\lambda e^{i\omega_2}B\right)x_0+\frac{1}{c_1c_2}\left(E+\lambda e^{i\omega_1}B\right)\left(E+\lambda e^{i\omega_2}B\right)y_0 ,$$

$$y_3=\frac{1}{c_3}\left(E+\lambda e^{i\omega_3}B\right)y_2+\frac{1}{c_3}e^{i\omega_3}Bx_2=\frac{1}{c_1c_2c_3}\left(E+\lambda e^{i\omega_1}B\right)\left(E+\lambda e^{i\omega_2}B\right)\left(E+\lambda e^{i\omega_3}B\right)y_0+$$

$$+\frac{1}{c_1c_2c_3}\frac{d}{d\lambda}\left(E+\lambda e^{i\omega_1}B\right)\left(E+\lambda e^{i\omega_2}B\right)\left(E+\lambda e^{i\omega_3}B\right)x_0 ,$$

....

$$y_{n-1}=\frac{1}{c_1\ldots c_{n-1}}\left(E+\lambda e^{i\omega_1}B\right)\left(E+\lambda e^{i\omega_2}B\right)\cdots\left(E+\lambda e^{i\omega_{n-1}}B\right)x_0+\frac{1}{c_1\ldots c_{n-1}}\frac{d}{d\lambda}\left(E+\lambda e^{i\omega_1}B\right)\times$$

$$\times\left(E+\lambda e^{i\omega_2}B\right)\cdots\left(E+\lambda e^{i\omega_{n-1}}B\right)x_0 .$$

Then

$$\frac{D_0y_0}{c_1\ldots c_{n-1}}+\frac{D_1\left(E+\lambda e^{i\omega_1}B\right)y_0+\frac{d}{d\lambda}D_1\left(E+\lambda e^{i\omega_1}B\right)x_0}{c_1\ldots c_{n-1}}+$$

$$+\frac{D_2\left(E+\lambda e^{i\omega_1}B\right)\left(E+\lambda e^{i\omega_2}B\right)y_0+\frac{d}{d\lambda}D_2\left(E+\lambda e^{i\omega_1}B\right)\left(E+\lambda e^{i\omega_2}B\right)x_0}{c_1\ldots c_{n-1}}+$$

$$+\frac{D_3\left(E+\lambda e^{i\omega_1}B\right)\cdots\left(E+\lambda e^{i\omega_3}B\right)y_0+\frac{d}{d\lambda}D_3\left(E+\lambda e^{i\omega_1}B\right)\cdots\left(E+\lambda e^{i\omega_3}B\right)x_0}{c_1\ldots c_{n-1}}+$$

$$+\cdots+\frac{D_{n-1}\left(E+\lambda e^{i\omega_1}B\right)\cdots\left(E+\lambda e^{i\omega_{n-1}}B\right)y_0}{c_1\ldots c_{n-1}}+\frac{D_{n-1}\frac{d}{d\lambda}\left(E+\lambda e^{i\omega_1}B\right)\cdots\left(E+\lambda e^{i\omega_{n-1}}B\right)x_0}{c_1\ldots c_{n-1}}=$$

$$=\frac{(E+\lambda K)\left(E+\lambda e^{i\omega_1}B\right)\ldots\left(E+\lambda e^{i\omega_{n-1}}B\right)y_0+\frac{d}{d\lambda}\left(E+\lambda e^{i\omega_1}B\right)\ldots\left(E+\lambda e^{i\omega_{n-1}}B\right)x_0}{c_1\cdots c_{n-1}}.$$

Thus

$$Y_0=\left(\sum_{i=0}^{n-1}\lambda^i A_i B^i+\lambda^{n-1}B^{n-1}+\lambda^n KB^{n-1}\right)y_0+\frac{1}{1!}\left(\frac{\partial}{\partial\lambda}\sum_{i=0}^{n-1}\lambda^i A_i B^i+\lambda^{n-1}B^{n-1}+\lambda^n KB^{n-1}\right)x_0.$$

We can do the same with regard to derivatives of all other orders.

So, we obtained the multiple completeness of a system of a polynomial pencil (1).

The proof of theorem 1 is completed.

Now consider the differential expression:

$$i_m u_m+p_1(x,\lambda)u^{(m-1)}+\cdots+p_{m-1}(x,\lambda)u'+\left[p_m(x,\lambda)-\lambda^n K\right]u+\lambda^{n-1}u, \qquad (11)$$

where $p_k(x,\lambda)=\sum_{j=0}^{r_k}\lambda^j q_{kj}(x)\ (k=1,2,\ldots,m)$, $q_{kj}(x)\ (j=0,1,\ldots,r_k;\ k=1,\ldots,m)$ are measurable essentially bounded complex functions determined on $[0,1]$, K is a self-adjoin completely continuous finite order operator.

Sign the boundary linearly independent conditions:

$$\sum_{k=1}^{m}\alpha_{jk}u^{(k-1)}(0)+\sum_{k=1}^{m}\beta_{jk}u^{(k-1)}(1)=0 \quad (j=1,2,\ldots,m), \qquad (12)$$

whose coefficients satisfy the relations:

$$\sum_{k=1}^{m}(-1)^k\left(\alpha_{jk}\overline{\alpha}_{l,m-k+1}-\beta_{jk}\beta_{,m-k+1l}\right)=0 \quad (j,l=1,2,\ldots,m). \qquad (13)$$

Denote by G a self-adjoined operator:

$$G=i_m u^{(m)}$$

in the space $L_2(0,1)$ with boundary conditions (12)-(13).

By $D_k\ (k=1,2,\ldots,m)$ we denote the k-fold differentiation operator acting in $L_2(0,1)$, determined on the set D_k of all the functions $u(x)\in L_2(0,1)$ such that $u^{(j)}(x)$ $(j=0,1,\ldots,k-1)$ are absolutely continuous, and $u^{(k)}(x)\in L_2(0,1)$.

Let

$$G=\sum_{j=1}^{\infty}\lambda_j(G)(\cdot,\varphi_j)\varphi_j$$

be a spectral expansion of the operator G, then

$$G_1=G+P,\quad G_1=\sum_{j=1}^{\infty}\left|\lambda_j(G)\right|(\cdot,\varphi_j)\varphi_j+P,,$$

where P is an orthogonal projector in $L_2(0,1)$ on the subspace $P(G)$.

By B_{kj} denote multiplying operator by the function $-q_{kj}(x)$. Obviously

$$M(\lambda)=G_1-\sum_{k=1}^{m}p_k(\lambda)D_{m-k}-\lambda^n KI+\lambda^{n-1}I,$$

where $D_0 = I$, $p_k(\lambda)=\sum_{j=0}^{r_k}\lambda^j\beta_{kj}\ (k=1,2,...,m-1)$, $p_m(\lambda)=\sum_{j=0}^{r_m}\lambda^j B_{mj}+P,\ (k=1,2,...,m-1)$.

The operator $G^{-1}(u)$ being the inverse to the operator G represented by means of a differential expression $i^m u^{(m)}$ with boundary initial conditions (12)-(13) is consequently, completely continuous self-adjoint finite order operator.

Using the arguments from [3] one can convince one-self that the above-mentioned conditions of theorem 1 are fulfilled. Thus, the completeness of the system of e.a. elements of the differential expression (11) holds.

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