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LOCALIZED APPROXIMATION OF FUNCTIONS WITH THE ANALYTICAL COMPLEX SPLINES IN DOMAINS WITH QUASI-CONFORM BOUNDARY

Abstract

In present paper the problem of localized approximation on the sets of the complex plain is considered. The localized problem of approximation of functions with the analytical complex splines in the domains with the quasi-conform boundary is investigated.

1. The main definitions, notations and auxiliary results.

Let G be a finite domain with the quasi-conform boundary $\Gamma = \partial G, 0 \in G$. $A(\overline{G})$ a class of functions analytical in G and continuous on closure $\overline{G}$.

Definition 1 [1]. *The image of circle on some quasi-conform mapping K of plain to itself is called quasi-conform curve or quasi-circle.*

Let $\overline{G} \subset Q$, where $Q = [a_1, a_1 + H] \times [b_1, b_1 + H]$ is the square with sides $H > 0$; $h_N = H/N$, where N is natural number; $x_k = a_1 + kh_N$; $y_j = b_1 + jh_N, k$, $j = 0,1,...,N$. It is obvious, that $Q = \bigcup\limits_{k,j}^{N-1} Q_{k,j}$, where

$$Q_{k,j} = \{z = x + iy : x \in [x_k, x_{k+1}],\ y \in [y_j, y_{j+1}]\}$$

square cells of division Q with step $h_N = H/N$.

Such division we'll denote by Δ_N, moreover we will omit index N in h_N and Δ_N, if the fixed division is considered and also we assume $G_N = \bigcup Q_{k,j}$ for all k, j for which $Q_{k,j} \cap G \neq \varnothing$.

Under the relation $A \prec B\ (A \geq 0, B \geq 0)$ we'll under stand inequality $A \leq CB$, where the constant $C > 0$ doesn't depend on A and B, and $A \approx B$, if $A \preceq B$ and $B \preceq A$, simultaneously.

Let $y(z)$ be a quasi-conform mapping of the complex plane $\mathbf{C}$ with respect to Γ [1]; for mapping $y(z)$ following properties are fulfilled (see [2], [3]):

1. $y(z)$ is an anti-quasi-conform mapping $y(\Gamma) = \Gamma, y(0) = \infty, y[y(z)] = z$;
2. for sufficient small fixed $\delta > 0$ in domain $\mathbf{C}_\delta = \mathbf{C}/\{O_\delta \cup O'_\delta\}$, where $O_\delta = \{z : |z| < \delta\}, O'_\delta = y(O_\delta)$, mapping $y = y(z)$ changes Euclidean lengths in the finite number time, moreover $|y_z| \preceq 1$, $|y_{\bar z}| \preceq 1$ almost for all $z \in \mathbf{C}_\delta$; $|y_z| \preceq |y(z)|^2$, $|y_{\bar z}| \preceq |y(z)|^2$ on $z \in O_\delta$; $|y_z| \preceq |z|^{-2}$; $|y_{\bar z}| \preceq |z|^{-2}$ on $z \in O'_\delta$.

One can continuously extend every function $f \in A(\overline{G})$ on $\mathbf{C}$, assuming $f(z) = f[y(z)]$ on $z \in C\overline{\mathbf{C}}$. It is known, that [5] for $f(z) \in A(\overline{G})$ the following integral representation is valid

$$f(z) = -\frac{1}{\pi} \iint_{CG} \frac{f[y(\zeta)]}{(\zeta - z)^2} y_{\bar\zeta} d\sigma_\zeta = -\frac{1}{\pi} \iint_{G} \frac{f(\zeta)}{(y(\zeta) - z)^2} y_{\bar\zeta} d\sigma_\zeta, \tag{1}$$

where $z \in G, \zeta = x + iy, d\sigma_\zeta = dxdy, CG = \mathbf{C}|C$.

In accordance with [6] let's construct on net domain G_N plane complex spline $S_A(z)$, interpolated function $f(z)$ (or its extension) on the tops of square $Q_{k,j}$, from G_N, assuming

$$S_\Delta(z) = a_1 + b_1 z + c_1 \bar{z} + d_1\left(z^2 - \bar{z}^2\right) \qquad \text{for} \qquad z \in Q_{k,j} CG_N, \tag{2}$$

where coefficients a_1, b_1, c_1, d_1 are determined from the interpolation condition of $f(z)$ at points $z_{k,j} = x_k + i y_j$. Note that, function $S_\Delta(z)$ continuous in G_N [6].

Definition 2 [4]. *We'll call the function*

$$S_\Delta(z) = -\frac{1}{\pi} \iint_{CG} \frac{S_\Delta[y(\zeta)]}{(\zeta - z)^2} y_{\bar{\zeta}} d\sigma_\zeta = \frac{1}{\pi} \iint_G \frac{S_\Delta[\zeta]}{(y(\zeta) - z)^2} y_{\bar{\zeta}} d\sigma_\zeta, \tag{3}$$

where $S_\Lambda(z)$ is a plane complex spline, analytical plane spline in domain G with the quasi-conform boundary.

If $S_\Delta(z)$ interpolates function $f(z)$ at nodes $z_{k,j}$, then we'll use notations $S_\Delta(f;z)$ for analytical spline (3).

In work [8] localized module of continuity was considered in the form

$$\omega_f^{z_0}(\delta, \eta) = \sup_{\substack{|z-t| \le \delta \\ z,t \in \Gamma_\eta(z_0)}} |f(z) - f(t)|, \tag{4}$$

where z_0 is a fixed point, $\delta, \eta > 0$ and $\Gamma_\eta(z_0) = \{z \in \Gamma, |z - z_0| \le \eta\}$.

Definition 3. *Let's denote by D the class of positive functions $\varphi(\delta, \eta)$, determined on $0 < \delta \le 2\eta < +\infty$ and such that*

a) $\varphi(\delta, \eta)$ *doesn't decrease on every argument;*

b) $\varphi(\delta, \eta) \cdot \delta^{-1}$ *doesn't increase on* δ;

c) $\exists \eta \in R_+ : \lim\limits_{\delta \to 0} \varphi(\delta, \eta) = 0$;

d) $\varphi(\delta, 2\eta) \preceq \varphi(\delta, \eta)$, *where in "$\preceq$" constant independent on δ and η.*

We'll denote by $C(\Gamma)$ the class of the functions are continuous on Γ.

Definition 4 [10]. *Let $\varphi \in D$. Assume*

$$H_{\varphi,\Gamma}^{z_0} = \left\{f \in C(\Gamma) : \omega_f(z_0, \delta, \eta) \le c\varphi(\delta, \eta), \forall \delta, \eta; 0 < \delta/2 \le \eta\right\},$$

where $\omega_f(z_0; \delta, \eta) = \delta \sup\limits_{\xi \ge \delta} \xi^{-1} \omega_f^{z_0}(\xi, \eta)$.

Definition 5. *Rectifiable Jordan curve Γ is called quasi-smooth, if for any pair of points z_1 and $z_2 \in \Gamma$ the length of its smaller part $\Gamma(z_1, z_2) \subset \Gamma$, lying between this points satisfies the following condition*

$$mes\Gamma(z_1, z_2) \preceq |z_1 - z_2|. \tag{5}$$

Lemma 1 [5]. *Let B be a bounded domain with quasi-conform bound Γ, $z_1, z_2 \in \Gamma$. There exists rectifiable arc $\overset{\smile}{z_1 z_2} \subset \overline{B}$, whose length $S\left(\overset{\smile}{z_1 z_2}\right)$ satisfies the inequality*

$$S\left(\overset{\smile}{z_1 z_2}\right) \le M_1 |z_1 - z_2|, \tag{6}$$

moreover, constant $M_1 > 1$ independent on B.

Lemma 2. *Let $f_1(z)$ be continuous in latticed domain G_N, and $S_\Delta(z)$ be a plane complex spline of form (3), interpolating $f_1(z)$ at nodes $\{z_{k,j}\}$. $z_0 \in G_N$ is a fixed point, $f_1 \in H_\varphi^{z_0}$. Then*

$$|f_1(z) - S_\Delta(z)| \preceq \varphi(h, h + |z - z_0|), \qquad where \qquad h = h_N. \tag{7}$$

Lemma 3. *Let plane complex spline $S_\Delta(z)$ interpolates continuous function $f_1(z)$ at the nodes of domain $\overline{G}_N$. $z_0 \in G_N$ is a fixed point and $f_1(z) \in H_\varphi^{z_0}$. Then for any points $z \in \overline{G}_N$ and $\zeta \in \overline{G}_N$ the estimation is valid:*

$$|S_\Delta(z) - S_\Delta(\zeta)| \preceq \varphi(h, h + |z - z_0|)|z - \zeta|. \tag{8}$$

It should use for proving lemmas 2 and 3 the scheme of proving of the estimation, which is in [4].

By virtue of the integral representation the following lemma is proved.

Lemma 4. *Let G be a finite domain with quasi-conform boundary Γ, $z_0 \in \Gamma$, $\varphi \in D$ and $f \in H_{\varphi,\Gamma}^{z_0} \cap A(\overline{G})$. Then on $\rho(z,\Gamma) = \min_{\zeta\in\Gamma}|z - \zeta|$*

$$|f''(z)| \preceq \frac{\varphi[\rho(z,\Gamma), \rho(z,\Gamma) + |z - z_0|]}{[\rho(z,\Gamma)]^2}, \quad z \in G. \tag{9}$$

Main results.

Theorem 1. *Let G be a finite domain with quasi-conform bound Γ, $0 \in G$, $z_0 \in \Gamma$, $\varphi \in D$ and $f(z) \in H_{\varphi,\Gamma}^{z_0} \cap A(\overline{G})$. Then in any point $z \in \Gamma = \partial G$ for analytical spline $S_{\Delta N}(f;z)$ the estimation is valid*

$$|f(z) - S_{\Delta N}(f;z)| \preceq \varphi(h, h + |z - z_0|)\left|\ln\frac{1}{h_N}\right|. \tag{10}$$

If Γ is a quasi-smooth curve, then the following relation has place:

$$|f(z) - S_{\Delta N}(f;z)| \preceq \varphi(h, h + |z - z_0|). \tag{11}$$

Proof of theorem 1. Let z_0 be fixed and z_1 be an arbitrary point of quasi-conform bound $\Gamma = \partial G$. $z \in G$ such point, that $\rho(z,\Gamma) < h/2$, $|z - z_1| = \rho(z,\Gamma)$. One can account, that $h = h_N < 1$. Let's denote $G_h = \{\zeta : \zeta \in \mathbf{C}|\overline{G}, |\zeta - z_1| \le mh\}$, where $m > 0$ is a constant independent on z_1 and h, $d = diam\overline{G}$. Let's consider circumference $C_h = \{\zeta : |\zeta - z_1| = mh\}$. Let us fix m such, that for $\zeta \in C_h \cap CG$ the condition is fulfilled: $|y(\zeta) - z_1| \ge 3h$. For mapping $y(z)$ condition $|y(\zeta) - z_1| \approx |\zeta - z_1|$ is fulfilled. So such m exists.

Now we fix sufficient small δ-neighbourhood O_δ of point $z = 0$. On mapping $y(z)$ its image will be some domain $O'_\delta = y(O_\delta)$. Let for O'_δ properties 1 and 2 of mapping $y(z)$ be fulfilled and

$$\rho(\Gamma, O'_\delta) = \inf_{z\in\Gamma, \zeta\in O'_\delta}|z - \zeta| > 3d. \tag{12}$$

It is clear that $\mathbf{C}|\overline{G} = [(\mathbf{C}|\overline{G}) \cap O_\eta(z_0)] \cup [(\mathbf{C}|\overline{G})|O_\eta(z_0)]$.

Assume

$$\Omega_1 = (\mathbf{C}|\overline{G}) \cap (O_\eta(z_0)|G_h) \cap [(\mathbf{C}|\overline{G})|O_\eta(z_0)],$$

$$\Omega_2=\left[\left(\mathbf{C}|\overline{G}\right)\cap\left(O_\eta(z_0)|G_h\right)\right]\cup\left[\left(\mathbf{C}|\overline{G}\right)|O_\eta(z_0)\right]|O'_\delta .$$

By virtue of (1) and (3) we get

$$f(z)-S_\Delta(f;z)=-\frac{1}{\pi}\iint\limits_{\mathbf{C}|G}\frac{f[y(\zeta)]-S_\Delta[y(\zeta)]}{(\zeta-z)^2}y_{\bar\zeta}d\sigma_\zeta=$$
$$=-\frac{1}{\pi}\iint\limits_{G_h}-\frac{1}{\pi}\iint\limits_{\Omega_2}-\frac{1}{\pi}\iint\limits_{O'_\delta}=J_1+J_2+J_3 \tag{13}$$

Let's estimate integral J_1. We have

$$J_1=-\frac{1}{\pi}\iint\limits_{G_h}\frac{f[y(\zeta)]-f(z_1)}{(\zeta-z)^2}y_{\bar\zeta}d\sigma_\zeta-\frac{1}{\pi}\iint\limits_{G_h}\frac{S_\Delta(z_1)-S_\Delta[y(\zeta)]}{(\zeta-z)^2}y_{\bar\zeta}d\sigma_\zeta+$$
$$+\frac{1}{\pi}\iint\limits_{G_h}\frac{[f(z_0)-S_\Delta(z_0)]}{(\zeta-z)^2}y_{\bar\zeta}d\sigma_\zeta=J_1^{(1)}+J_1^{(2)}+J_1^{(3)}. \tag{14}$$

We'll estimate each integral $J_1^{(i)},\ i=1,2,3$.

Let's consider for all $u\in G$ the function

$$F_1(u)=\int\limits_{\breve{OU}}[f(t)-f(z_1)]dt, \tag{15}$$

where integral is taken on any rectifiable arc $\breve{OU}\subset\overline{G}$. It is clear that $F_1(u)\in A(\overline{G})$.

We'll carry out its QC extension $\widetilde{F}_1(u)$ in expensed plane.

Assume

$$\frac{(f\circ y)(\zeta)-f(z_1)}{(\zeta-z_1)^2}y_{\bar\zeta}=N(\zeta). \tag{16}$$

Using lemma 1, the property of module continuity $\omega_f^{z_0}(\delta,\eta)$ and taking into account, that $\varphi(\delta,\delta)/\delta^\mu$ almost decreases, we have

$$\left|J_1^{(1)}\right|\preceq\varphi\left(h,h+|z_1-z_0|\right). \tag{17}$$

Now, let's estimate integral $J_1^{(2)}$. By virtue of lemma 3 we have

$$\left|J_1^{(2)}\right|\preceq\varphi\left(h,h+|z_1-z_0|\right). \tag{18}$$

Let's estimate integral $J_1^{(3)}$. For that we consider the function

$$f_1(z)=\begin{cases}z-z_1, \text{ for } z\in\overline{G}\,;\\ y(z)-z_1, \text{ for } z\in\mathbf{C}|\overline{G}\,.\end{cases}$$

One can apply the Borel-Pompey formula to the function $f_1(z)$. The following is true

$$f_1(z)=-\frac{1}{\pi}\iint\limits_{G_h}\frac{f_{1\bar\zeta}}{(\zeta-z)}d\sigma_\zeta+\frac{1}{2\pi i}\int\limits_{C_h}\frac{f_1(\zeta)}{(\zeta-z)}d\zeta\,. \tag{19}$$

From (19) and using, that $|y(\zeta)-z_1|\approx|\zeta-z_1|$ we have

$$\left|\frac{1}{\pi}\iint\limits_{G_h}\frac{y_{\bar\zeta}}{(\zeta-z)^2}d\sigma_\zeta\right|\preceq 1. \tag{20}$$

Using (20) and lemma 2, we get

$$\left|J_1^{(3)}\right| \preceq \left|f(z_1) - S_\Delta(z_1)\right| \left|\iint\limits_{G_h} \frac{y_{\bar\zeta}}{(\zeta - z)^2} d\sigma_\zeta\right| \preceq \varphi\left(h, h + |z_1 - z_0|\right). \tag{21}$$

By virtue of (14), (17), (18), (21) we are

$$|J_1| \preceq \varphi\left(h; h + |z_1 - z_0|\right). \tag{22}$$

Let's estimate integrals J_2 and J_3. Denote $L = \max\limits_{z\in\Gamma, \zeta\in O'_\delta} |z - \zeta|$, $l = \rho(\Gamma, O'_\delta)$. By virtue of lemma 2 we have:

$$\begin{aligned}|J_2| &\preceq \varphi\left(h; h + |z_1 - z_0|\right) \iint\limits_{\Omega_1 | O'_\delta} \frac{d\sigma_\zeta}{|\zeta - z|^2} \preceq \varphi\left(h; h + |z_1 - z_0|\right) \int\limits_0^{2\pi} d\alpha \times \\ &\times \int\limits_{(m-1/2)h}^{L} \frac{dt}{d} \preceq \varphi\left(h; h + |z_1 - z_0|\right) \cdot \ln \frac{2L}{(2m-1)h} \preceq \varphi\left(h; h + |z_1 - z_0|\right) \cdot \ln \frac{1}{h}.\end{aligned} \tag{23}$$

Using lemma 2 we have also:

$$|J_3| \preceq \varphi\left(h; h + |z_1 - z_0|\right) \iint\limits_{O'_\delta} \frac{d\sigma_\zeta}{|\zeta|^2 |\zeta - z|^2} \preceq \varphi\left(h; h + |z_1 - z_0|\right). \tag{24}$$

Taking into account (13), (22), (23), (24) we will get for all $z \in G$, $|z - z_1| < h/2$

$$\left|f(z) - S_\Delta(f; z)\right| \preceq \varphi\left(h; h + |z_1 - z_0|\right) \cdot \ln \frac{1}{h}, \tag{25}$$

where the constant in relation "$\preceq$" independent on z_0, z and h. Hence estimation follows.

Now we will show, that when Γ is a quasi-conform curve, then estimation (11) is valid. Proving estimation (10) note that only estimation (23) is unique, which contains $\ln \frac{1}{h}$. Let's include following notations [4]. Let $Q_1 = \{\zeta : \zeta \in G | O'_\delta ; |\zeta - z_1| \geq 3h\}$, $m > 0$ we can choose such that $y(\Omega_2) \subset Q_1$. And also we denote

$$Q_2 = \left\{\zeta \in Q_1 : \rho(\zeta, \Gamma) \geq \left(3h|z_1 - \zeta|\right)^{1/2}\right\},\ Q_3 = Q_1 | Q_2,$$
$$Q_3^{(j)} = \left\{\zeta \in Q_3 : 3 \cdot 2^j h \leq |\zeta - z_1| \leq 3 \cdot 2^{j+1} h\right\},\ j = 0, 1, \ldots, K,$$

where $Q_3^{(K)} \neq \varnothing$, and for all $j > K$, $Q_3^{(j)} = \varnothing$.

It is obvious, that on $\zeta \in Q_1$ relation $|y(\zeta) - z| \approx |\zeta - z_1|$ is valid. It is proved that, $Q_{k,j} \subset G_N$ such square that $\rho(Q_{k,j}, \Gamma) \geq h$, then for any point $z \in Q_{k,j}$ the following estimation is valid:

$$\left|f(z) - S_\Delta(z)\right| \preceq \frac{h\varphi\left(h, |z - z_0| + h\right)}{\rho(z, \Gamma)}. \tag{26}$$

Using (26) of lemma 4 and relation $|y(\zeta) - z| \approx |\zeta - z_1|$ we obtain

$$\begin{aligned}|J_2| &= \left| -\frac{1}{\pi} \iint\limits_{y(\Omega_2)} \frac{f(\zeta) - S_\Delta(\zeta)}{(y(\zeta) - z)^2} y_{\bar\zeta} d\sigma_\zeta \right| \leq \frac{1}{\pi} \iint\limits_{Q_2} \frac{|f(\zeta) - S_\Delta(\zeta)|}{|y(\zeta) - z|^2} |y_{\bar\zeta}| d\sigma_\zeta + \\ &+ \frac{1}{\pi} \iint\limits_{Q_3} \frac{|f(\zeta) - S_\Delta(\zeta)|}{|y(\zeta) - z|^2} |y_{\bar\zeta}| d\sigma_\zeta = J_2^{(1)} + J_2^{(2)},\end{aligned} \tag{27}$$

$$J_2^{(1)} \preceq h^{1/2} \varphi\left(h, h + |\zeta - z_1| + |z_1 - z_0|\right) \iint\limits_{Q_2} \frac{d\sigma_\zeta}{|\zeta - z_1|^{5/2}}. \tag{28}$$

Let's consider the two possible cases:

1) $|\zeta - z_1| < |z_1 - z_0|$;

2) $|z_1 - z_0| \le |\zeta - z_1|$.

Let case 1) has place. Then we have:

$$J_2^1 \preceq h^{1/2}\varphi(h, h + |z_1 - z_0|)\iint_{\Omega_2}\frac{d\sigma_\zeta}{|\zeta - z_1|^{5/2}} \preceq \varphi(h, h + |z_1 - z_0|).$$

Now let case 2) has place. Then taking into account, that $\varphi(\delta, \delta)\cdot\delta^{-\mu}$ almost decreases, we obtain

$$J_2^{(1)} \preceq h^{1/2}\varphi(|\zeta - z_1|, |\zeta - z_1|)\iint_{\Omega_2}\frac{d\sigma_\zeta}{|\zeta - z_1|^{5/2}} \preceq h^{1/2}\frac{\varphi(h,h)}{h^\mu}\iint_{\Omega_2}\frac{d\sigma_\zeta}{|\zeta - z_1|^{5/2-\mu}} \preceq \tag{29}$$
$$\preceq \varphi(h, h + |z_1 - z_0|).$$

Thus

$$J_2^{(1)} \preceq \varphi(h, h + |z_1 - z_0|). \tag{30}$$

Now let's estimate integral $J_2^{(2)}$. Using lemma 2 and reasoning by analogy to the foregoing one we obtain

$$J_2^{(2)} \preceq \varphi(h, h + |z_1 - z_0|)\sum_{j=0}^{K}\iint_{Q_3^{(j)}}\frac{d\sigma_\zeta}{|\zeta - z_1|^2} \preceq \varphi(h, h + |z_1 - z_0|)\cdot\sum_{j=0}^{K}\frac{mes Q_3^{(j)}}{2^{2j}h^2}. \tag{31}$$

The following estimation is valid (see [4])

$$\sum_{j=0}^{K}\frac{mes Q_3^{(j)}}{2^{2j}h^2} \preceq \sum_{j=0}^{K}2^{-j/2}. \tag{32}$$

Putting estimation (32) into (31), finally we obtain

$$J_2^{(2)} \preceq \varphi(h, h + |z_1 - z_0|). \tag{33}$$

From (27), (30), (33) it follows, that

$$|J_2| \preceq \varphi(h, h + |z_1 - z_0|).$$

So theorem 1 has been proved. In this theorem, particularly, the V.I. Belov's and I.V. Strelkovsky's [4] result is contained.

References

[1]. Альфорс Л. *Лекции по квазиконформным отображениям.* М., Мир, 1969, 133с.

[2]. Белый В.И., Гридасова И.В. *Исследование граничных свойств одного интегрального преобразования.* Теория отображений и приближение функций. Киев, Наук.думка, 1983, с. 20-29.

[3]. Белый В.И., Гридасова И.В. *некоторые сингулярные интегралы с квазиконформной границей.* Мат.заметки, 1985(37), №4, с. 497-506.

[4]. Белый В.И., Стрелковский И.В. *Приближение функций аналитическими комплексными сплайнами в областях с квазиконформной границей.* Укр.мат.журн., 1988, т.40, №5, с. 563-568.

[5]. Белый В.И. *Конформные отображения и приближение функций в областях с квазиконформной границей.* Мат.сб., 1977, т.102, №3, с. 331-361.

[6]. Opfer G., Puri M. *Complex planar splines.* J. Approx. Theory, 1981, v. 31, №4, P.383-402.

[7]. Завьялов Ю.С., Квасов Б.И., Мирошниченко В.Л. *Методы сплайн функций.* М., Наука, 1980, 352с.

[8]. Мамедханов Дж.И., Салаев В.В. *О новых классах функций связанных с локальной структурой особых интегралов и некоторые аппроксимации в них.* Тезисы докладов Всесоюзного симпозиума по теории аппроксимации. Уфа, 1980, с. 91-92.

[9]. Дзядык В.К. *Введение в теорию равномерного приближения функций полиномами.* Наука, 1977, 512с.

[10]. Мусаев А.А. *О некоторых вопросах локализованной аппроксимации в комплексной плоскости.* Автореферат диссертации на соискание ученой степени кандидата физико-математических наук. Баку, 1987.

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