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KULIEV A.D., MAMEDOV F.I.

ON THE NONLINEAR WEIGHT ANALOGUE OF THE LANDIS-GERVER'S TYPE MEAN VALUE THEOREM AND ITS APPLICATIONS TO QUASI-LINEAR EQUATIONS

Abstract

The weight analogy of Lagrange's mean value theorem is obtained. Removable sets for the solutions of equation (1) in Hölderian class C^{α} are studied.

In [1] the authors have proved a generalization of Lagrange's classic mean value theorem for the multidimensional case. Later on, this theorem turned out to be fruitful for its applicability to various problems of quality theory of elliptic and parabolic equations ([see 2,3,4,5,6]).

We prove the weight analogy of the mentioned theorem adapted to investigation of quality properties of the degenerate quasi-linear equation

$$\frac{\partial}{\partial x_i}\left(\omega(x)|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right)=0. \tag{1}$$

In the paper we also study removable sets for the solution of equation (1), in C^{α}.

Let $\omega(x)$ be a positive measurable function satisfying the doubling condition: for concentric balls B_R^x and B_{2R}^x of R and $2R$ radius, there exists such a constant γ

$$\omega(B_R^x)\geq\gamma\omega(B_{2R}^x),$$

where for the measurable sets E $\omega(E)$ means $\int\limits_E\omega(y)dy$.

The main result of the paper is the following theorem.

Let $\nabla u=\left\{\frac{\partial u}{dx_1},...,\frac{\partial u}{dx_n}\right\}$ be a gradient of the function u, $|\nabla u|^2=\left(\frac{\partial u}{\partial x_1}\right)^2+...+\left(\frac{\partial u}{\partial x_n}\right)^2$, $\operatorname*{osc}\limits_D u=\sup\limits_D u-\inf\limits_D u$.

Theorem 1. *Let $2\leq p<\infty$, D be a bounded domain in the spherical layer $R\leq|x|\leq 2R$, $u(x)\in C^2(D)$. Then there exists a piece-wise smooth surface Σ, dividing in D the spheres $|x|=R$ and $|x|=2R$, that*

$$\int\limits_{\Sigma}\omega(x)|\nabla u|^{p-2}\left|\frac{\partial u}{\partial n}\right|d\tau\leq K\frac{\left(\operatorname*{osc}\limits_D u\right)^{p-1}\omega(D)}{R^p}, \tag{2}$$

where $K>0$ is some constant depending on the dimension of the space and on p, $\frac{\partial u}{\partial n}$ means a derivative in normal to Σ.

Prove the next Lemma, which we shall use in the proof of Theorem 1.

Denote by O_f a set of singular points $\{x\in D:\nabla f=0\}$ of the function f in D.

Lemma 1. *Let $2 \le p < \infty$, $n \ge 2$ $D \subset R^n$ be a bounded domain, $f(x) \in C^2(D)$. Then for any $\varepsilon > 0$ there exists a finite number of balls $\{B_{r_\nu}^{x_\nu}\}$, $\nu = 1,2,\dots,N$ which cover O_f and such that if we denote by S_ν the surface of ν-th ball, then*

$$\sum_{\nu=1}^{N} \int_{S_\nu} \omega(x) |\nabla f|^{p-1} ds < \varepsilon. \tag{3}$$

Proof. Decompose O_f into two parts: $O_f = O'_f \cup O''_f$, where O'_f is a set of points O_f for which $\nabla^2 f \ne 0$, O''_f is a set of points O_f for which $\nabla^2 f = 0$.

The set O'_f has n-dimensional Lebesque measure equal to zero, as on the known implicit function theorem, the O'_f lies on a denumerable number of surfaces of dimension $n-1$. If we use the absolute continuity of integral

$$\omega(G) = \int_G \omega(x) dx$$

with respect to Lebesque measure G and above said we get that the set O'_f may be included into the set G for which $\omega(G) < \eta$, $\eta > 0$ will be choose later. Let for each point $x \in O'_f$ there exist such r_x that $B_{r_x}^x$ and $B_{6r_x}^x$ are contained in $G \subset D$. Then

$$\int_{5r_x}^{6r_x} dr \int_{S_r^x} \omega(\sigma) d\sigma \le \omega\left(B_{6r_x}^x\right)$$

therefore there exists such $5r_x \le t \le 6r_x$ that

$$r_x \int_{S_t^x} \omega(\sigma) d\sigma \le \omega\left(B_{6r_x}^x\right).$$

Then

$$\int_{S_t^x} \omega(\sigma) |\nabla f|^{p-1} d\sigma \le C^{p-1} t^{p-1} \int_{S_t^x} \omega(\sigma) d\sigma \le (6C)^{p-1} r_x^{p-2} \left(r_x \int_{S_t^x} \omega(\sigma) d\sigma \right) \le$$
$$\le (6C)^{p-1} r_x^{p-2} \omega\left(B_{6r_x}^x\right) \le (6C)^{p-1} \alpha^{p-2} \gamma^{-3} \omega\left(B_{r_x}^x\right) \le C_0 \omega\left(B_{t/5}^x\right), \tag{4}$$

where $C = \sup_D |\nabla^2 f|$, $\alpha = diam G$, $C_0 = (6C)^{p-1} \alpha^{p-2} \gamma^{-3}$.

Now by a Babach process ([4], p.126), from the ball system $\{B_{t/5}^x\}$ we choose such a denumerable number of not-intersecting balls $\{B_{t_\nu/5}^{x_\nu}\}$ $\nu = 1,2,\dots,N$ that the ball of five times greater radius $\{B_{t_\nu}^{x_\nu}\}$ cover the whole O'_f set. We again denote these balls by $\{B_{t_\nu}^{x_\nu}\}$ $\nu = 1,2,\dots,N$ and their surface by S'_ν. Then by virtue of (4)

$$\sum_{\nu=1}^{\infty} \int_{S'_\nu} \omega(\sigma) |\nabla f|^{p-1} d\sigma \le C_0 \omega(G) < C_0 \eta. \tag{5}$$

Now let $x \in O''_f$. Then

$$\int_{r_x}^{2r_x} d\tau \int_{S_\tau^x} \omega(\sigma) d\sigma \le \omega\left(B_{2r_x}^x\right)$$

Therefore there exists such $r_x \le t \le 2r_x$ that

$$r_x \int_{S_t^x} \omega(\sigma) d\sigma \le \omega\left(B_{2r_x}^x\right).$$

Assign arbitrary $\eta > 0$. By virtue of that $|\nabla f|_{S_t^x} \le \eta \cdot t$, for sufficiently small t we have

$$\int_{S_t^x} \omega(\sigma) |\nabla f|^{p-1} d\sigma \le \eta^{p-1} t^{p-1} \int_{S_t^x} \omega(\sigma) d\sigma \le (2\eta)^{p-1} r_x^{p-2} \left(r_x \int_{S_t^x} \omega(\sigma) d\sigma \right) \le$$

$$\le (2\eta)^{p-1} r_x^{p-2} \omega\left(B_{2r_x}^x\right) \le (6C)^{p-1} \le \alpha^{p-2} (2\eta)^{p-1} \gamma^{-1} \omega\left(B_{t/5}^x\right) \le \eta C_1 \omega\left(B_{t/5}^x\right). \tag{6}$$

Again by means of Banach process and by virtue of (6), we get

$$\sum_{\nu=1} \int_{S_\nu''} \omega(\sigma) |\nabla f|^{p-1} d\sigma \le \eta \cdot C_1 \omega(D), \tag{7}$$

where S_ν'' is the surface of balls in the second covering.

Combining the spherical surfaces S_ν' and S_ν'' we get that the open balls system cover the closed set O_f. Then a finite subcovering may be choosing from it. Let they be the balls $B_1, B_2, ..., B_x$ and their surfaces is $S_1, S_2, ... S_N$.

We get from inequalities (3) and (5)

$$\sum_{\nu=1}^{N} \int_{S_\nu} |\nabla f|^{p-1} \omega(\sigma) d\sigma \le \left[C_1 \omega(\overline{D}) + C_0\right] \cdot \eta .$$

Put now $\varepsilon = \left[C_1 \omega(\overline{D}) + C_0\right] \cdot \eta$, lemma 1 is proved.

Proof of Theorem 1. Following [2], assume

$$\varepsilon = \frac{\omega(D) \left(\underset{D}{osc}\, u\right)^{p-1}}{R^p} \tag{8}$$

and according to Lemma 1 we'll find the balls $B_1, B_2, ..., B_N$ for given ε and exclude them from the domain D. Put $D^* = D \setminus \bigcup_{\nu=1}^{N} B_\nu$. Intersect D^* with a closed spherical layer

$$R\left(1 + \frac{1}{4}\right) \le |x| \le R\left(1 + \frac{5}{4}\right).$$

We denote the intersection by D'. We can assume that the function $u(x)$ is defined in some δ-vicinity D_δ' of the set D'. Take $\delta < \frac{R}{4}$ so that

$$\underset{D_\delta'}{osc}\, u \le 2 \underset{D}{osc}\, u . \tag{9}$$

On a closed set D' we have $\nabla f \ne 0$. Consider on D_δ' the equation system

$$\frac{dx}{dt} = u_x . \tag{10}$$

Let S a such form surface that it touches to field direction at any his point, then

$$\int_S \omega(x) |\nabla u|^{p-2} \left|\frac{\partial u}{\partial n}\right| d\sigma = 0 ,$$

since $\frac{\partial u}{\partial n}$ is identically equal to zero at S.

We shall use it in constructing the needed surface of Σ. Tubular surfaces whose generators will be the trajectories of the system (10) constitute the basis of Σ.

They will add nothing to the integral we are interested in. These surfaces will have the form of thin tubes that cover D'. Then we shall put partitions to some of these tubes.

Let's construct tubes. Denote by E the intersection of D' with sphere $|x| = R\left(1+\frac{3}{4}\right)$.

Let N be a set of points E, where field direction of system (10) touches the sphere $|x| = R\left(1+\frac{3}{4}\right)$. Cover N with such an open on the sphere $|x| = R\left(1+\frac{3}{4}\right)$ set F that

$$\int_F \omega(x)|\nabla u|^{p-2}\left|\frac{\partial u}{\partial n}\right| d\sigma \le \frac{\omega(D)\left(\operatorname{osc}_D u\right)^{p-1}}{R^p}. \tag{11}$$

It will be possible if on N $\frac{\partial u}{\partial R} \equiv 0$.

Put $E' = E \setminus F$. Cover E' on the sphere by a finite number of open domains with piece-wise smooth boundaries. We shall call them cells. We shall control their diameters in estimation of integrals that we need. The surface remarked by the trajectories lying in the ball $|x| \le \frac{7}{4}R$ and passing through the bounds of cells we shall call tube.

So, we obtained a finite number of tubes. The tube is called open if not intersecting this tube one can join by a broken line the point of its corresponding cell with a spherical layer $\frac{5}{4}R - \delta < |x| < \frac{7}{4}R$. Choose the diameters of cells so small that the trajectory beams passing through each cell, could differ no more than $\frac{\delta}{2n}$.

By choose of cells diameters the tubes will be contained in

$$\frac{5}{4}R - \frac{\delta}{2} < |x| < \frac{5}{4}R.$$

Let also the cell diameter be chosen so small that the surface that is orthogonal to one trajectory of the tube intersect the other trajectories of the tube at an angle more than $\frac{\pi}{4}$.

Cut off the open tube by the hypersurface in the place where it has been imbedded into the layer

$$\frac{5}{4}R - \frac{\delta}{2} < |x| < \frac{5}{4}R$$

at first time so that the edges of this tube be embedded into this layer.

Denote these cut off tubes by $T_1, T_2, .., T_S$. If each open tube is divided with a partition, then a set-theoretical sum of closed tubes, tubes $T_1, T_2, ..., T_S$, their partitions, spheres $S_1, S_2, ..., S_N$ and the set F on the sphere $|x| = \frac{7}{4}R$ divides the spheres $|x| = R$ and $|x| = 2R$.

Note that $\int\limits_S \omega|\nabla u|^{p-2}\left|\frac{\partial u}{\partial n}\right| d\sigma$ along the surface of each tube equals to zero, since $\frac{\partial u}{\partial n}$ identically equals to zero.

Now we have to choose partitions so that the integral $\int\limits_S \omega|\nabla u|^{p-2}\left|\frac{\partial u}{\partial n}\right| d\sigma$ was of the desired value. Denote by U_i the domain bounded by T_i, with corresponding cell and hypersurface cutting off this tube. We have $U_i \cap U_j = \varnothing$ and therefore

$$\sum_{i=1}^{m} \omega(U_i) < 2\omega(D). \tag{12}$$

Consider a tube T_i and corresponding domain U_i. Choose any trajectory on this tube. Denote it by L_i. The length $\mu_i L_i$ of the curve L_i satisfies the inequality

$$\mu_i L_i \geq \frac{R}{2}. \tag{13}$$

On L_i introduce a parameter in l-length of the arc counted from cell. By $\sigma_i(l)$ denote the cross-section by U_i hypersurface passing through the point, corresponding to l, and orthogonal to the trajectory L_i at this point. Let the diameter of cells be so small that

$$\int\limits_{L_i} dl \int\limits_{\sigma_i(l)} \omega(x) d\sigma < 2\omega(U_i). \tag{14}$$

Then by Chebyshev inequality a set H of points $l \in L_i$ where

$$\int\limits_{\sigma_i(l)} \omega(x) d\sigma > \frac{8}{R}\omega(U_i)$$

satisfies the inequality $\mu_i H < \frac{R}{4}$ and hence by virtue of (13) for $E = L_i \setminus H$ it is valid

$$\mu_1 E > \frac{R}{4}, \tag{15}$$

and

$$\int\limits_{\sigma_i(l)} \omega(x) d\sigma < \frac{8}{R}\omega(U_i) \text{ for } l \in E. \tag{16}$$

At the points of the curve L_i the derivative $\frac{\partial u}{\partial l}$ preserves its sign, and therefore

$$\int\limits_E \left|\frac{\partial u}{\partial l}\right| dl \leq \int\limits_{L_i} \left|\frac{\partial u}{\partial l}\right| dl \leq \operatorname*{osc}_{D'_\delta} u.$$

Hence, by using (15) and a mean value theorem for one variable function we find that there exists $l_0 \in E$

$$\left|\frac{\partial u}{\partial l}\right|\Bigg|_{l=l_0} \leq \frac{4}{R}\operatorname*{osc}_{D'_\delta} u.$$

But on the other hand

$$\left|\frac{\partial u}{\partial l}\right|\Bigg|_{l=l_0} = |\nabla u|\big|_{l=l_0}.$$

So

$$\left|\nabla u\right|^{p-1}\Big|_{l=l_0} \le \left(\frac{4}{R}\operatorname*{osc}_{D} u\right)^{p-1}.$$

Together with (16) it gives

$$\left|\nabla u\right|^{p-1}\Big|_{l=l_0} \int\limits_{\sigma_i(l_0)} \omega(x) d\sigma \le \frac{8}{R}\frac{4^{p-1}}{R^{p-1}}\,\omega(U_i)\cdot\left(\operatorname*{osc}_{D} u\right)^{p-1}.$$

Now, let the diameter of cells be still so small that

$$\int\limits_{\sigma_i(l_0)} \omega(x)\left|\nabla u\right|^{p-1} d\sigma \le \frac{16\cdot 4^{p-1}}{R^{p-1}}\,\omega(U_i)\cdot\left(\operatorname*{osc}_{D} u\right)^{p-1}$$

(we can do it, since the derivatives $\frac{\partial u}{\partial x_i}$ are uniformly continuous). Therefore, according to (12)

$$\sum_{i=1}^{S} \int\limits_{\sigma_i(l_0)} \omega(x)\left|\nabla u\right|^{p-1} d\sigma \le \frac{32\cdot 4^{p-1}}{R^{p-1}}\,\omega(D)\cdot\left(\operatorname*{osc}_{D} u\right)^{p-1}. \qquad (17)$$

Now by Σ we denote a set-theoretic sum of all open tubes, all through tubes T_i, all $\sigma_i(l_0)$, all spheres S_i and sets F on the sphere $|x| = \frac{7}{4}R$.

Then, we get by (3), (9), (11), (17)

$$\int\limits_{\Sigma} \omega(x)\left|\nabla u\right|^{p-2}\left|\frac{\partial u}{\partial n}\right| d\sigma \le K\frac{\omega(D)\cdot\left(\operatorname*{osc}_{D} u\right)^{p-1}}{R^{p}}.$$

The theorem 1 is proved.

2. Now apply the mean value theorem to the study of removable sets of solutions of equation (1). Following [7,4] a removable set is understood as follows.

Let in a bounded domain $D \subset R^n$ be given a function $u(x)$ from some class F. Assume that $u(x)$ is the solution of equation (1) outside of some compactum $E \subset D$. If for any function $u \in F$ from condition $u(x)\big|_{\partial D} = 0$ follows $u(x) \equiv 0$ in D, then we say that the set E is removable for the class F (see [7]). We study the removable sets for class the solutions of equation (1) in $D|E$, with Hölder continuity order α $(0 \le \alpha \le 1)$. Following [7] we get the definition of Karatheodory measure of the set.

Let E be a compact subset from R^n, $h(r)$ be such a continuous increasing function $h(0) = 0$ and $\mu(G)$ be positive denumerable-additive measure, determined on all Borell subsets of the compact E. Cover the set E by denumerable set of balls B_ν, of radius $r_\nu \le \rho$, $0 < \rho < \infty$ with a center at the point x_ν. Put

$$\Lambda_\mu^h(\rho) = \inf \sum_{\nu} h(r_\nu)\mu(B_\nu),$$

where the infimum is taken in all coverings.

Limit

$$\Lambda_\mu^h(E) = \lim_{\rho \to 0} \Lambda_\mu^h(\rho)$$

obviously exists. In the case when $d\mu = dx$, $h(r) = r^{s-n}$, $0 < s < \infty$, $\Lambda^h(E)$ gives a Hausdorfian measure - $mes_S(E)$ (see [7]).

Let ω be a function described at the beginning of the paper, $0<\alpha\leq 1$, $C^{\alpha}(D)$ be a class of functions $u:D\to(-\infty,+\infty)$ for which

$$\sup_{\substack{x\neq x',\\ x\in D,\\ x'\in D}}\frac{|u(x)-u(x')|}{|x-x'|^{\alpha}}<\infty .$$

The function $u\in C^{1}(\overline{D})$ we shall call a solution of the equation

$$\frac{\partial}{\partial x_i}\left(\omega|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right)=0 \quad \text{in} \quad D,$$

if for any function $\varphi\in C^{1}(D)$ it holds the identity

$$\sum_{i=1}^{n}\int_{D}\omega|\nabla u|^{p-2}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_i}dx=\int_{\partial D}\omega|\nabla u|^{p-2}\varphi(x)\frac{\partial u}{\partial n}d\sigma ,$$

where $\dfrac{\partial}{\partial n}$ denotes a derivative in external normal to ∂D.

Theorem 2. *Let $0<\alpha\leq 1$, $u(x)\in C^{\alpha}(D)\cap C^{1}(D|E)$ be the solution of equation (1) outside of the compactum $E\subset D$. For the set E to be removable, it is sufficient that*

$$\Lambda_{\mu}^{h}(E)=0,$$

where $h(r)=r^{-p+(p-1)\alpha}$, $d\mu=\omega dx$.

Proof. Cover the set E by a denumerable system of balls $\{B_{\nu}\}_{\nu=1,2,\dots}$, so that

$$\sum_{\nu}r_{\nu}^{-p+(p-1)\alpha}\omega(B_{\nu})<\varepsilon ,$$

$\varepsilon>0$ is any number.

Along with the balls consider the covering $B_{2\nu}$ with concentric B_{ν} balls of radius $2r_{\nu}$. For each ν by theorem 1 there exists such a piece-wise smooth surface γ_{ν} dividing the spheres $\partial B_{2\nu}$ and ∂B_{ν} that

$$\int_{\gamma_{\nu}}\omega|\nabla u|^{p-2}\left|\frac{\partial u}{\partial n}\right|dS\leq C\frac{\omega(B_{2\nu}|B_{\nu})\left(\operatorname*{osc}_{D}u\right)^{p-1}}{r_{\nu}^{p}}\leq cr_{\nu}^{-p+(p-1)\alpha}\omega(B_{2\nu}). \tag{18}$$

The interior of the surface γ_{ν} denote by Γ_{ν}. Then $\Gamma=\bigcup\Gamma_{\nu}\supset B_{\nu}\supset E$. Put $\sigma_{\nu}=\Gamma\cap\gamma_{\nu}$.

It follows from (18)

$$\int_{\sigma_{\nu}}\omega|\nabla u|^{p-2}\left|\frac{\partial u}{\partial n}\right|dS\leq cr_{\nu}^{-p+(p-1)\alpha}\omega(B_{2\nu}).$$

Then from equation (1) we get

$$\int_{D|\Gamma}\omega|\nabla u|^{p}dx=\int_{\partial\Gamma}\omega(x)|\nabla u|^{p-2}|u|\left|\frac{\partial u}{\partial n}\right|ds\leq$$

$$M\sum_{\nu}\int_{\gamma_{\nu}}\omega(x)|\nabla u|^{p-2}\left|\frac{\partial u}{dn}\right|ds\leq M\sum_{\nu}cr_{\nu}^{-p+(p-1)\alpha}\omega(B_{2\nu})\leq$$

$$\leq\varepsilon M,\quad M=\sup_{\Omega}|u|.$$

As ε is arbitrary we get that

$$\int_{D|E}\omega(x)|\nabla u|^{p}dx=0,$$

hence $u \equiv 0$ in D.

Corollary. *Let* $u(x) \in C^{\alpha}(D) \cap C^{1}(D|E)$, $(0 < \alpha \leq 1)$ *be the solution of the equation*

$$\frac{\partial}{\partial x_i}\left(\omega|\nabla u|^{p-2}\frac{du}{\partial x_i}=0\right), \quad 2 \leq p < \infty$$

in $D|E$. *Then for removability of the set* E *in* C^{α} *it is sufficient*

$$mes_{n-p+(p-1)\alpha} E = 0 .$$

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Kuliev A.D.
Institute of Mathematics and Mechanics of AS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-47-20 (off.).

Mamedov F.I.
Azerbaijan Technique University.
25, H.Javid av., 370073, Baku, Azerbaijan.

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