MAMEDOV I.T., GULIEV A.F.

STABILIZATION OF SOLUTIONS OF THE FIRST BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION

Abstract

The paper deals with the problem on stabilization for $t \to \infty$ of solutions of the first boundary value problem for the heat equation in unbounded domains, which represent figures of rotation with respect to time axis.

Introduction.

Let $\mathbf{R}_{n+1}$ be $(n+1)$-dimensional Euclidean space of points $(x,t)=(x_1,...,x_n,t)$. Let's consider in $\mathbf{R}_{n+1}$ unbounded domain $D=\{(x,t):|x|^2<t\alpha(t),\ 1<t<\infty\}$, where $\alpha(t)$ is continuous non-decreasing on $(1,\infty)$ function, moreover there exist the positive constants K_1 and K_2 such that for sufficient large t

$$\alpha'(t)\le\frac{K_1}{t}, \tag{1}$$

$$\alpha(t)\le K_2 \ln\ln\ln t. \tag{2}$$

Let further $\Gamma_1=\{(x,t):|x|^2<\alpha(1),\ t=1\}$, $\Gamma_2=\{(x,t):|x|^2=t\alpha(t),\quad 1\le t<\infty\}$. Purpose of this paper is to prove the fact of stabilization for $t\to\infty$ of the solution of the first boundary value problem

$$\Delta u-u_t=0,\ (x,t)\in D;\ u|_{\Gamma_1}=\varphi(x),\ \ u|_{\Gamma_2}=0, \tag{3}$$

where $\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is Laplacian, $\varphi\in C(\overline{\Gamma}_1)$.

Let's note that the investigations of stabilization problems of solutions of the Cauchy problem for the second order parabolic equations begin from A.N. Tikhonov's classic work [1]. With this connection we point also to works [2-3] and the monography [4]. Concerning the study of behavior of solutions of parabolic equations in unbounded domains, we mention the works [5-9].

1^0. Denotations, definitions and auxiliary confirmations.

Let E be B-set in $\mathbf{R}_{n+1}$. We will call the measure μ admissible on E if

$$\int_E G(x-y,t-\tau)d\mu(y,\tau)\le 1 \ \text{ for } (x,t)\notin E,$$

where $G(x,t)=\begin{cases} t^{-\frac{n}{2}}\exp\left[-\frac{|x|^2}{4t}\right], & \text{if } t>0; \\ 0, & \text{if } t\le 0. \end{cases}$

Number $p(E)=\sup\mu(E)$, where the least upper bound is taken by all admissible measures, is called heat capacity of set E.

Everywhere further if Ω is a domain in $\mathbf{R}_{n+1}$, then $\partial\Omega$ means its boundary.

Let's denote for natural m

$$A_m = \left\{ (x,t) : e^{-(m+1)\frac{n}{2}} \le G(x,t) \le e^{-\frac{mn}{2}} \right\}, \quad H_m = A_m \setminus D .$$

It is not difficult to see that set A_m is given by the inequalities

$$A_m = \left\{ (x,t) : 2nt \ln \frac{e^m}{t} \le |x|^2 \le 2nt \ln \frac{e^{m+1}}{t}, \; 0 \le t \le e^{m+1} \right\}.$$

Let further z_m $(m = 1,2,...,)$ be time coordinate of points of intersection of the level surface $G(x,t) = e^{-\frac{mn}{2}}$ by Γ_2. Thus, z_m is the root of the equation

$$\alpha(z_m) = 2n \ln \frac{e^m}{z_m}. \tag{4}$$

Now let us show that for sufficiently large m equation (4) really has the root on interval $\left(e^{\frac{m}{2}}, e^m \right)$. With this purpose let's consider the function $f(z) = \alpha(z) - 2n \ln \frac{e^m}{z}$ for $z \in \left[e^{\frac{m}{2}}, e^m \right]$. We have $f(e^m) = \alpha(e^m) > 0$. Further, taking in to account (2) we obtain $f\left(e^{\frac{m}{2}} \right) \le K_2 \ln \ln \frac{m}{2} - nm < 0$, if only m is sufficiently large. Hence the existence of the root of equation in the requested interval follows. If there exist more that one such root, then we'll denote their greatest lower bound by z_m.

Let's prove that the sequences $\{z_m\}$ increases. Assume the contrary that for some m $z_{m+1} \le z_m$. Since $z_m = e^m \cdot \exp\left[-\frac{\alpha(z_m)}{2n} \right]$, then by virtue of our assumption and non-decreasing of function $\alpha(z)$

$$1 \ge \frac{z_{m+1}}{z_m} = e \cdot \exp\left[-\frac{\alpha(z_{m+1}) - \alpha(z_m)}{2n} \right] \ge e ,$$

that is impossible. Thus, for all natural m

$$z_m < z_{m+1} \le e\, z_m . \tag{5}$$

Let's consider the sequence $v_m = e^{-m} z_m$. It's not difficult to see that for all m $v_{m+1} \le v_m$. Really

$$\frac{v_m}{v_{m+1}} = \exp\left[-\frac{1}{2n} \left((\alpha_m) - \alpha(z_{m+1}) \right) \right] \ge 1 .$$

Therefore, there exists limit $v_0 = \lim_{m \to \infty} v_m$. From condition (1) it follows that if t_1 and t_2 are sufficiently large and $t_1 > t_2$, then

$$\alpha(t_1) - \alpha(t_2) = \int_{t_2}^{t_1} \alpha'(z) dz \le K_1 \ln \frac{t_1}{t_2} .$$

So for sufficiently large m

$$\frac{z_{m+1}}{z_m} \geq e \cdot \left(\frac{z_{m+1}}{z_m}\right)^{-\frac{K_1}{2n}},$$

i.e.

$$\frac{z_{m+1}}{z_m} \geq 1+\beta, \text{ where } \beta = e^{\frac{2n}{2n+K_1}} - 1. \tag{6}$$

Lemma 1. *If with respect to function $\alpha(t)$ the conditions (1)-(2) are fulfilled, then for sufficiently large m*

$$C_1(n)\left(z_m \ln\frac{1}{\nu_m}\right)^{\frac{n}{2}} \leq p(H_m) \leq C_2(n)\left(z_m \ln\frac{1}{\nu_m}\right)^{\frac{n}{2}}. \tag{7}$$

Here and further the note $C(\cdots)$ means that the positive constant C depends only on the contain of the brackets.

Proof. Consider the case $\nu_0 = 0$. The case $\nu_0 > 0$ is considered by analogy. First let's find the value of $t \in (0, z_{m+1})$ for which the function $2nt \ln\frac{e^{m+1}}{t}$ reaches its maximum. It is not difficult to see, that this value of t is equal to e^m. On the other hand, since $\nu_0 = 0$, then for sufficiently large m $z_{m+1} < e^m$. So, if for $R > 0$, $t_1 < t_2$ to denote the cylinder $\{(x,t): |x| < R,\ t_1 < t < t_2\}$ by $C_R^{t_1,t_2}$, then

$$H_m \subset C^{0,z_{m+1}}_{\sqrt{2nz_{m+1}\ln\frac{e^{m+1}}{z_{m+1}}}} \subset C^{0,2nz_{m+1}\ln\frac{e^{m+1}}{z_{m+1}}}_{\sqrt{2nz_{m+1}\ln\frac{e^{m+1}}{z_{m+1}}}}$$

for sufficiently large m. Thus, by [10]

$$p(H_m) \leq p\left(C^{0,2nz_{m+1}\ln\frac{e^{m+1}}{z_{m+1}}}_{\sqrt{2nz_{m+1}\ln\frac{e^{m+1}}{z_{m+1}}}}\right) = C_3(n)\left(z_{m+1}\ln\frac{e^{m+1}}{z_{m+1}}\right)^{\frac{n}{2}}.$$

Using (5) we obtain

$$p(H_m) \leq C_3 e^{\frac{n}{2}}\left[z_m\left(\ln\frac{1}{\nu_m}+1\right)\right]^{\frac{n}{2}} \leq (2e)^{\frac{n}{2}} C_3\left(z_m \ln\frac{1}{\nu_m}\right)^{\frac{n}{2}},$$

if only m is sufficiently large. Thus, the upper estimation in (7) has been proved with $C_2 = (2e)^{\frac{n}{2}} C_3$. Let's estimate now $p(H_m)$ from below. Let for natural m $S_m = \left\{(x,t): G(x,t) = e^{-\frac{mn}{2}}\right\} \cap \{(x,t): 1 \leq t \leq z_m\}$. Consider the integral

$$I = \int_S G(x-y, t-\tau)\, dS_{(y,\tau)},$$

where $dS_{(y,\tau)}$ is surface Lebesgue measure on S_m. Let's go over from integration over S_m to integration over the projection $\widetilde{S}_m$ of surface S_m on hyperplane $t=1$. Let (x',t') and (y',τ') be images of points (x,t) and (y,τ) correspondingly for such projection. We have

$$dS_{(y,\tau)}=\sqrt{1+\sum_{i=1}^{n}\left(\frac{\partial\tau}{\partial y_i}\right)^2}dy'=\sqrt{1+\left(\frac{d\tau}{d|y|}\right)^2\sum_{i=1}^{n}\left(\frac{\partial|y|}{\partial y_i}\right)^2}dy'=\sqrt{1+\left(\frac{d\tau}{d|y|}\right)^2}dy'.$$

But on the other hand

$$\frac{d\tau}{d|y|}=1\Big/\frac{d|y|}{d\tau};\quad \frac{d|y|}{d\tau}=\frac{\sqrt{2n}}{2}\frac{\ln\frac{e^m}{\tau}-1}{\sqrt{\tau\ln\frac{e^m}{\tau}}}\geq\frac{\sqrt{2n}}{4}\sqrt{\frac{2n\frac{e^m}{z_m}}{z_m}},$$

if only m is sufficiently large.

Therefore, for sufficiently large m

$$dS_{(y,\tau)}\leq C_4(n)\sqrt{\frac{z_m}{\ln\frac{1}{\nu_m}}}dy',$$

so for $t'>\tau'$ we obtain

$$I\leq C_4\sqrt{\frac{z_m}{\ln\frac{1}{\nu_m}}}\int_{S_m}(t'-\tau')^{\frac{n}{2}}\exp\left[-\frac{|x'-y'|^2}{4(t'-\tau')}\right]dy'\leq$$

$$\leq C_5(n)\sqrt{\frac{z_m}{\ln\frac{1}{\nu_m}}}\left(\int_{-\infty}^{\infty}e^{-t^2}dt\right)^n=C_6(n)\sqrt{\frac{z_m}{\ln\frac{1}{\nu_m}}}.$$

Hence it follows that for sufficiently large m the measure $\frac{1}{C_6}\sqrt{\frac{\ln\frac{1}{\nu_m}}{z_m}}dS_{(y,\tau)}$ is admissible. Then if S_m' is the cylindric surface $\left\{(x,t):|x|^2=2nz_m\ln\frac{1}{\nu_m},1\leq t\leq z_m\right\}$ and m is sufficiently large, then

$$p(H_m)\geq\frac{1}{C_6}\sqrt{\frac{\ln\frac{1}{\nu_m}}{z_m}}mes(S_m)\geq\frac{1}{C_6}\sqrt{\frac{\ln\frac{1}{\nu_m}}{z_m}}mes(S_m')=$$

$$=C_7(n)\sqrt{\frac{\ln\frac{1}{\nu_m}}{z_m}}(z_m-1)\left(z_m\ln\frac{1}{\nu_m}\right)^{\frac{n-1}{2}}\geq\frac{C_7}{2}\left(z_m\ln\frac{1}{\nu_m}\right)^{\frac{n}{2}},$$

where mes is n-dimensional Lebesgue surface measure. Thus, we have obtained the lower estimation in (7) with $C_1=\frac{C_7}{2}$. The lemma has been proved.

Corollary. *Let for natural* m $H_m'=\{(x,t):z_m\leq t\leq z_{m+1}\}\cap H_{m+3}$. *Then if the conditions of the lemma are fulfilled, then for sufficiently large* m

$$p(H_m')\geq C_8(n,K_1)p(H_m).\qquad(8)$$

Really, being restricted by the case $\nu_0=0$ and acting as in proof of the lemma, we are persuaded that

$$\int_{\Gamma_m} G(x-y,t-\tau)dS_{(y,\tau)} \le C_9(n)\sqrt{\frac{z_m}{\ln\frac{1}{\nu_m}}},$$

if only m is sufficiently large. Here $\Gamma_m = \left\{(x,t): G(x,t) = e^{-\frac{(m+3)n}{2}},\ z_m \le t \le z_{m+1}\right\}$. So for sufficiently large m the measure $\frac{1}{C_9}\sqrt{\frac{\ln\frac{1}{\nu_m}}{z_m}}dS_{(y,\tau)}$ is admissible. Hence we conclude that if Γ'_m is the cylindric surface $\left\{(x,t): |x|^2 = 2nz_{m+1}\ln\frac{e^{m+3}}{z_{m+1}},\ z_m \le t \le z_{m+1}\right\}$ and m is sufficiently large, then

$$p(H'_m) \ge \frac{1}{C_9}\sqrt{\frac{\ln\frac{1}{\nu_m}}{z_m}}mes(\Gamma_m) \ge \frac{1}{C_9}\sqrt{\frac{\ln\frac{1}{\nu_m}}{z_m}}mes(\Gamma'_m) =$$

$$= C_{10}(n)\sqrt{\frac{\ln\frac{1}{\nu_m}}{z_m}}(z_{m+1}-z_m)\left(z_{m+1}\ln\frac{e^{m+3}}{z_{m+1}}\right)^{\frac{n-1}{2}}.$$

Taking into account (5) and (6) we obtain for sufficiently large m

$$p(H'_m) \ge C_{10}\beta\sqrt{\frac{\ln\frac{1}{\nu_m}}{z_m}}z_m\left(z_m\left(\ln\frac{e^m}{z_m}-1\right)\right)^{\frac{n-1}{2}} \ge \frac{C_{10}\beta}{2}\left(z_m\ln\frac{1}{\nu_m}\right)^{\frac{n}{2}}.$$

From the last inequality and (7) immediately the requested estimation (8) follows with $C_8 = \frac{C_{10}\beta}{2C_2}$. The corollary has been proved.

2^0. Theorem on increasing of positive solutions.

Let for natural $m > 1$ $D_{m,1} = D \cap \{(x,t): z_m < t < z_{m+3}\}$, $D_{m,2} = D \cap \{(x,t): z_{m+2} < t < z_{m+3}\}$.

Theorem 1. *Let in D non-negative solution $u(x,t)$ of the first boundary value problem (3) be defined and with respect to function $\alpha(t)$ the conditions (1)-(2) are fulfilled. Then there exist the positive constants $\eta = \eta(n,K_1)$ and $b_0 = b_0(n,K_1,K_2)$ such that if m is sufficiently large, then*

$$\sup_{D_{m,1}} u \ge \left[1 + \eta(\ln m)^{-b_0} e^{-\frac{mn}{2}} p(H_m)\right]\sup_{D_{m,2}} u. \qquad (9)$$

Proof. Let's be restricted again by consideration of the case $\nu_0 = 0$. Let H'_m have the same sense that in the corollary to lemma 1, and the measure μ_m on H'_m is such that for $m > 1$

$$U_m(x,t) = \int_{H'_m} G(x-y,t-\tau)d\mu_m(y,\tau) \le 1, \quad (x,t) \notin H'_m, \qquad (10)$$

$$\mu_m(H'_m) \geq \frac{1}{2} p(H'_m). \tag{11}$$

Let's consider in $D_{m,1}$, the auxiliary function

$$W_m(x,t) = V_m[1 - U_m(x,t)] - u(x,t),$$

where $V_m = \sup\limits_{D_{m,1}} u$. According to (10) $W_m(x,t) \geq 0$ for $(x,t) \in \Gamma(D_{m,1})$, where $\Gamma(D_{m,1})$ is parabolic boundary of domain $D_{m,1}$ (see [10]). By the maximum principle $W_m(x,t) \geq 0$ for $(x,t) \in D_{m,1}$, and particularly

$$\sup_{D_{m,2}} u = \sup_{T_{m+2}} W_m \leq \sup_{T_{m+2}} [V_m(1 - U_m)] = V_m\left(1 - \inf_{T_{m+2}} U_m\right), \tag{12}$$

where for natural m $T_m = D \cap \{(x,t) : t = z_m\}$.

Let $(x,t) \in T_{m+2}$, $(y,\tau) \in H'_m$. We have by virtue of (5)

$$G(x-y, t-\tau) \geq \left(\frac{z_{m+2}}{z_m}\tau\right)^{-\frac{n}{2}} \exp\left[-\frac{|x-y|^2}{4(t-\tau)}\right] \geq$$

$$\geq e^{-n} \tau^{-\frac{n}{2}} \exp\left[-\frac{|x-y|^2}{4(t-\tau)}\right] = e^{-n} \tau^{-\frac{n}{2}} i_1 . \tag{13}$$

But on the other hand

$$i_1 = \left(\exp\left[-\frac{|y|^2}{4(t-\tau)}\right]\right)^{\frac{|x-y|^2}{|y|^2}} \geq \exp\left[-\frac{|y|^2}{4(t-\tau)}\right]\left(\exp\left[-\frac{|y|^2}{4(t-\tau)}\right]\right)^{\frac{|x|^2+2|x||y|}{|y|^2}} . \tag{14}$$

Since $|x|^2 \leq z_{m+2}\alpha(z_{m+2}) = 2nz_{m+2} \ln\frac{e^{m+2}}{z_{m+2}} \leq 2nz_{m+2} \ln\frac{e^{m+3}}{z_m}$, and $|y|^2 \geq 2nz_m \ln\frac{e^{m+3}}{z_m}$, then according to (5)

$$\frac{|x|^2 + 2|x||y|}{|y|^2} \leq \frac{z_{m+2}}{z_m} + 2\sqrt{\frac{z_{m+2}}{z_m}} \leq e^2 + 2e < 16 .$$

So from (14) we conclude that

$$i_1 \geq \exp\left[-\frac{|y|^2}{4(t-\tau)}\right] \cdot \exp\left[-\frac{4|y|^2}{t-\tau}\right] = \exp\left[-\frac{|y|^2}{4(t-\tau)}\right] i_2 . \tag{15}$$

Further taking into account (2), (4)-(6) for sufficiently large m we have

$$\frac{4|y|^2}{t-\tau} \leq \frac{8nz_{m+2} \ln\frac{e^{m+4}}{z_{m+2}}}{z_{m+2} - z_{m+1}} \leq \frac{8n}{\beta}\frac{z_{m+2}}{z_{m+1}} \ln\frac{e^{m+4}}{z_{m+2}} \leq \frac{4e}{\beta}(\alpha(z_m) + 4) \leq \frac{8e}{\beta}\alpha(z_m) \leq \frac{8K_2 e}{\beta} \ln\ln\ln z_m .$$

Remembering that $z_m < e^m$ and denoting $\frac{8K_2 e}{\beta}$ by b_1, we obtain

$$i_2 \geq (\ln m)^{-b_1},$$

if only m is sufficiently large. Taking now into account the last estimation in (15), from (13)-(14) we conclude that for sufficiently large m

$$G(x-y,t-\tau)\geq e^{-n}(\ln m)^{-b_1}\tau^{-\frac{n}{2}}\exp\left[-\frac{|y|^2}{4(t-\tau)}\right]. \tag{16}$$

By analogy we deduce

$$\exp\left[-\frac{|y|^2}{4(t-\tau)}\right]=\exp\left[-\frac{|y|^2}{4\tau}\right]\cdot\left(\exp\left[-\frac{|y|^2}{4\tau}\right]\right)^{\frac{\tau}{t-\tau}-1}=\exp\left[-\frac{|y|^2}{4\tau}\right]i_3 \tag{17}$$

and further

$$\frac{\tau}{t-\tau}-1=\frac{2\tau-t}{t-\tau}\leq\frac{2\tau}{t-\tau}\leq\frac{2z_{m+1}}{z_{m+2}-z_{m+1}}\leq\frac{2}{\beta}.$$

Therefore

$$i_3\geq\exp\left[-\frac{|y|^2}{2\beta\tau}\right]. \tag{18}$$

On the other hand for sufficiently large m

$$\frac{|y|^2}{2\beta\tau}\leq\frac{n}{\beta}\ln\frac{e^{m+4}}{\tau}\leq\frac{n}{\beta}\left(\ln\frac{e^m}{z_m}+4\right)\leq\frac{2n}{\beta}\ln\frac{e^m}{z_m}=\frac{1}{\beta}\alpha(z_m)\leq$$

$$\leq\frac{K_2}{\beta}\ln\ln\ln z_m\leq\frac{K_2}{\beta}\ln\ln m.$$

Therefore from (18) it follows that

$$i_3\geq(\ln m)^{-b_2},$$

if only m is sufficiently large, and $b_2=\frac{K_2}{\beta}$. The last inequality with (16)-(17) implies for sufficiently large m the estimation

$$G(x-y,t-\tau)\geq e^{-n}(\ln m)^{-b_0}G(y,\tau), \tag{19}$$

where $b_0=b_1+b_2$. Since $(y,\tau)\in H_m'$, then

$$G(y,\tau)\geq e^{-\frac{(m+4)n}{2}}.$$

Thus from (19) finally we obtain

$$G(x-y,t-\tau)\geq e^{-3n}(\ln m)^{-b_0}e^{-\frac{mn}{2}}. \tag{20}$$

From (20), (11) and (8) it follows that for sufficiently large m

$$\inf_{T_{m+2}}U_m\geq\frac{e^{-3n}}{2}(\ln m)^{-b_0}e^{-\frac{mn}{2}}p(H_m')\geq\frac{C_8e^{-3n}}{2}(\ln m)^{-b_0}e^{-\frac{mn}{2}}p(H_m).$$

Let's denote $\frac{C_8e^{-3n}}{2}$ by η. Then from (12) we conclude that for sufficiently large m

$$\sup_{D_{m,2}}u\leq V_m\left(1-\eta(\ln m)^{-b_0}e^{-\frac{mn}{2}}p(H_m)\right).$$

Hence the requested inequality (9) follows. The theorem has been proved.

Let's denote for natural $m>1$ $\sup_{T_m}u$ by M_m.

Corollary. *If the conditions of the theorem are fulfilled and m is sufficiently large, then*

$$M_m \geq \left[1+\eta(\ln m)^{-b_0} e^{-\frac{mn}{2}} p(H_m)\right] M_{m+2}.$$

For proof it is sufficient to use the maximum principle in (9).

3^0. Theorems on stabilization of solutions.

Let for $R>1$ $M(R)=\sup\limits_{D(R)}|u|$, where $D(R)=D\cap\{(x,t):t=R\}$.

Denote for $\tau>1$ the set $\left\{(x,t): e^{-(\tau+1)\frac{n}{2}} \leq G(x,t) \leq e^{-\frac{\tau n}{2}}\right\} \setminus D$ by $H(\tau)$.

Theorem 2. *Let in D the solution $u(x,t)$ of the first boundary-value problem (3) and with respect to function $\alpha(t)$ the conditions (1)-(2) be fulfilled. Then there exists the positive constant $\eta_0=\eta_0(n,K_1,K_2)$ such that if R is sufficient great, then*

$$M(R) \leq \sup_{\Gamma_1}|\varphi| \exp\left[-\eta_0 \int_2^{\ln R} (\ln\tau)^{-b_0} e^{-\frac{\tau n}{2}} p(H(\tau)) d\tau\right]. \tag{21}$$

Proof. Let's denote by i_0 the minimal natural number such that for $m \geq 2i_0$ the confirmation of theorem 1 takes place and let j be such natural number, that

$$z_{2j} \leq R < z_{2(j+1)}. \tag{22}$$

We consider R so large, that $j>i_0$. Let $D^+=\{(x,t):u(x,t)>0\}$. Without losing of generality we may assume that $D^+ \neq \varnothing$. In contrary, all below-reduced reasonings should be applied to function $-u(x,t)$. Using sequently the corollary to theorem 1, we obtain

$$M^+_{2i_0} \geq (1+\eta\gamma(2i_0))M^+_{2(i_0+1)} \geq \cdots \geq \prod_{i=i_0}^{j-1}(1+\eta\gamma(2i))M^+_{2j}, \tag{23}$$

where for natural $i>1$ $\gamma(i)=(\ln i)^{-b_0} e^{-i\frac{n}{2}} p(H_i)$, $M^+_i=\sup\limits_{T^+_i} u$, $T^+_i=D^+\cap\{(x,t):t=z_i\}$.

By the principle of maximum and (22) we conclude, that

$$M^+_{2j} \geq \mathcal{M}^+(R)=\sup_{D(R)\cap D^+} u, \quad M^+_{2i_0} \leq \mathcal{A}=\sup_{\Gamma_1}|\varphi|.$$

Therefore, from (23) it follows that

$$\mathcal{A} \geq \prod_{i=i_0}^{j-1}(1+\eta\gamma(2i))\mathcal{M}^+(R),$$

and

$$\mathcal{M}^+(R) \leq \mathcal{A}\exp\left[-\sum_{i=i_0}^{j-1}\ln(1+\eta\gamma(2i))\right]. \tag{24}$$

It is not difficult to see that (see the proof of lemma 1) for all natural i

$$H_i \subset C^{0,e^{i+1}}_{0,\sqrt{2ne^i}}.$$

So, according to [10]

$$p(H_i) \leq C_{11}(n) e^{\frac{in}{2}},$$

and therefore for all natural $i > 1$

$$\gamma(i) \le C_{12}(n, b_0).$$

Let's consider now for $t \in (0, C_{12}]$ the function $\psi(t) = \ln(1+\eta t)/t$. Since $\lim_{t\to 0+} \psi(t) = \eta$, then there exists $\delta \in (0, C_{12})$ such that $\psi(t) \ge \frac{\eta}{2}$ for $t \in (0, \delta)$. On the other hand $\psi(t) \ge \frac{\ln(1+\eta\delta)}{C_{12}}$ for $t \in [\delta, C_{12}]$.

Thus

$$\psi(t) \ge \eta_1(n, K_1, b_0) = \min\left\{\frac{\eta}{2}, \frac{\ln(1+\eta\delta)}{C_{12}}\right\}.$$

Taking into account (24) we obtain

$$\mathcal{M}^+(R) \le \mathcal{A} \exp\left[-\eta_1 \sum_{i=i_0}^{j-1} \gamma(2i)\right]. \tag{25}$$

Now let's make one remark. The confirmation of the theorem has a contain sense only in the case when

$$\int_2^\infty (\ln \tau)^{-b_0} e^{-\frac{\tau n}{2}} p(H(\tau)) d\tau = \infty$$

or in the equivalent form

$$\sum_{i=2}^\infty \gamma(i) = \infty$$

(without losing of generality we consider that the sequence $\gamma(i)$ doesn't increase for $i = 2, \ldots, $). Therefore we can suppose that at least one of the following series diverges

$$\sum_{i=1}^\infty \gamma(2i) \quad \text{or} \quad \sum_{i=1}^\infty \gamma(2i+1).$$

Suppose for definiteness, that

$$\sum_{i=1}^\infty \gamma(2i) = \infty. \tag{26}$$

Further we have

$$\sum_{i=i_0}^{j-1} \gamma(2i) = \sum_{i=1}^{j-1} \gamma(2i) - \sum_{i=1}^{i_0-1} \gamma(2i) = \frac{1}{2}\sum_{i=1}^{j-1} \gamma(2i) + \frac{1}{2}\left(\sum_{i=1}^{j-1} \gamma(2i) - 2\sum_{i=1}^{i_0-1} \gamma(2i)\right).$$

By virtue of (26) for sufficiently large j (i.e. for sufficiently large R)

$$\sum_{i=1}^{j-1} \gamma(2i) \ge 2\sum_{i=1}^{i_0-1} \gamma(2i).$$

Taking into account the last inequality in (25) we obtain

$$\mathcal{M}^+(R) \le \mathcal{A} \exp\left[-\frac{\eta_1}{2} \sum_{i=1}^{j-1} \gamma(2i)\right]. \tag{27}$$

Then if for $\tau > 1$ $\bar{\gamma}(\tau) = (\ln \tau)^{-b_0} e^{-\frac{\tau n}{2}} p(H(\tau))$, then from (27) it follows

$$\mathcal{M}^+(R) \le \mathcal{A} \exp\left[-\frac{\eta_1}{2} \int_1^j \bar{\gamma}(2\tau) d\tau\right]. \tag{28}$$

On the other hand according to (22)

$$\int_1^j \bar{\gamma}(2\tau)d\tau \geq \int_1^{\frac{1}{2}\ln R-1} \bar{\gamma}(2\tau)d\tau = \int_1^{\frac{1}{2}\ln R} \bar{\gamma}(2\tau)d\tau - \int_{\frac{1}{2}\ln R-1}^{\frac{1}{2}\ln R} \bar{\gamma}(2\tau)d\tau \geq \frac{1}{2}\int_1^{\frac{1}{2}\ln R} \bar{\gamma}(2\tau)d\tau = \frac{1}{4}\int_2^{\ln R} \bar{\gamma}(\tau)d\tau,$$

if only R is sufficiently large. Therefore from (28) we conclude

$$\mathcal{M}^+(R) \leq \mathcal{A} \exp\left[-\frac{\eta_1}{8}\int_2^{\ln R} \bar{\gamma}(\tau)d\tau\right]. \tag{29}$$

By quite analogy we can show, that if $D^- = \{(x,t): u(x,t) < 0\}$, $D^- \neq \varnothing$ and $\mathcal{M}^-(R) = \sup\limits_{D(R)\cap D^-} |u|$, then for sufficiently large R

$$\mathcal{M}^-(R) \leq \mathcal{A} \exp\left[-\frac{\eta_1}{8}\int_2^{\ln R} \bar{\gamma}(\tau)d\tau\right]. \tag{30}$$

Now from (29)-(30) the requested estimation (21) with $\eta_0 = \frac{\eta_1}{8}$ follows. The theorem has been proved.

Let's consider now the case when the function $\alpha(t)$ in the definition of domain D for sufficiently large t is represented in the form $\alpha(t) = K_2 \ln\ln\ln t$.

Theorem 3. *Let in abovepointed domain D the solution $u(x,t)$ of the first boundary value problem (3) be defined. Then there exist the positive constants $\xi = \xi(n, K_2)$ and $d = d(n, K_2)$ such that if R is sufficiently large, then*

$$\mathcal{M}(R) \leq \sup_{\Gamma_1}|\varphi| \exp\left[-\xi \frac{\ln R}{(\ln\ln R)^d}\right]. \tag{31}$$

Proof. Let's denote for sufficiently large τ by $z(\tau)$ the greatest lower bound of roots of the equation

$$\alpha(z(\tau)) = 2n \ln \frac{e^\tau}{z(\tau)}, \tag{32}$$

which are arranged on interval $\left(e^{\frac{\tau}{2}}, e^\tau\right)$ and $\nu(\tau) = e^{-\tau} z(\tau)$. Acting as in the proof of lemma 1, we obtain that for sufficient large τ

$$p(H(\tau)) \geq C_1 \left(z(\tau) \ln \frac{1}{\nu(\tau)}\right)^{\frac{n}{2}}. \tag{33}$$

From (32) we have

$$z(\tau) = e^\tau \exp\left[-\frac{\alpha(z(\tau))}{2n}\right] = e^\tau \exp\left[-\frac{K_2}{2n}\ln\ln\ln z(\tau)\right] \geq e^\tau (\ln \tau)^{-\frac{K_2}{2n}}.$$

Moreover, for our domain D $\lim\limits_{\tau\to\infty} \nu(\tau) = 0$, so $\ln \frac{1}{\nu(\tau)} \geq 1$, if only τ is sufficiently large. Therefore from (33) we conclude that

$$p(H(\tau)) \geq C_1 e^{\frac{\tau n}{2}} (\ln \tau)^{-\frac{K_2}{4}}, \ \tau \geq \tau_0(n, K_2). \tag{34}$$

Using (34) in (21) and denoting $b_0 + \frac{K_2}{4}$ by d we obtain

$$\mathcal{M}(R)\le \mathcal{A}\exp\left[-\eta_0 C_1\int_{\tau_0}^{\ln R}(\ln\tau)^{-d}\,d\tau\right]. \tag{35}$$

Now it is sufficient to consider that for sufficiently large R $\ln R-\tau_0\ge\dfrac{1}{2\ln R}$ and from (35) the requested estimation (31) with $\xi=\dfrac{\eta_0 C_1}{2}$ follows. The theorem has been proved.

Therefore, in the case of the considered domain D the solution $u(x,t)$ decreases at infinity with "almost" degree velocity.

References

[1]. Тихонов А.Н. *Теоремы единственности для уравнения теплопроводности.* Матем.сборник, 1935, т.42, №2, с.199-216.

[2]. Ильин А.М. *О поведении решения задачи Коши для параболического уравнения при неограниченном возрастании времени.* Успехи матем. наук, 1961, т. XVI, вып.2(98), с.115-121.

[3]. Репников В.Д., Эйдельман С.Д. *Новое доказательство теоремы о стабилизации решения задачи Коши для уравнения теплопроводности.* Матем.сборник, 1967, т.73, №1, с.155-159.

[4]. Эйдельман С.Д. *Параболические системы.* М., «Наука», 1964, 443с.

[5]. Дринь Р.Я. *Стабилизация решений задачи Коши для систем параболических псевдодифференциальных уравнений с негладкими символами.* В сб.: «Інтеграли перетворення та їх застосування до крайових задач», Киев, 1997, №14, с.88-101.

[6]. Репников В.Д. *О стабилизации решений параболических уравнений с дивергентной эллиптической частью.* Дифф.уравнения, 1995, т.31, №1, с.114-122.

[7]. Ротанов Н.Е. *Стабилизация пространственно-временных статистических решений параболического уравнения.* Вестник Челяб. Ун-та, сер.мат.-мех., 1991, №1, с.64-73.

[8]. Ильин А.М. *Об одном достаточном условии стабилизации решения параболического уравнения.* Матем.заметки, 1985, т.37, №6, с.851-856.

[9]. Черемных Ю.Н. *Об асимптотике решений параболических уравнений.* Изв. АН СССР, 1959, т.23, №6, с.913-924.

[10]. Ландис Е.М. *Уравнения второго порядка эллиптического и параболического типов.* М., «Наука», 1971, 288с.

Mamedov I.T., Guliyev A.F.
Institute of Mathematics and Mechanics of AS Azerbaijan.
9, F.Agayev str., 370141, Baku, Azerbaijan.
Tel.: 39-39-24 (off.).

Received February 14, 2000; Revised August 23, 2000.
Translated by Soltanova S.M.