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TOLERANCE TOPOLOGIES AND COMMUTATORS ON LATTICES

Abstract

Our aim is to present a generalization of Schweigert's tolerances definable by commutators on lattices; we connect this with a tolerance topology on the considering lattice. Then a necessary condition is given for strengthening of our tolerance to a congruence.

Let L be a complete lattice with the least element 0 and the greatest element 1. A tolerance of L is a reflexive and symmetric sublattice of $L \times L$. General information on tolerance relations of algebras may be found in [4]. Of course every congruence is a transitive tolerance and vice versa.

Let T be a set of tolerances on L which is the base of some filter [2] on $L \times L$, i.e. for $\tau_1, \tau_2 \in \mathrm{T}$ there exists some $\tau \in \mathrm{T}$ such that $\tau \subseteq \tau_1$ and $\tau \subseteq \tau_2$ (and, of course, $\varnothing \notin \mathrm{T}$; here this condition is satisfied automatically). The topology on L which T naturally gives rise to will be called a T -tolerance topology. It is a simple matter to see that under this topology the lattice operations $\wedge$ and $\vee$ are continuous and thus L is a T -tolerance topological lattice. Of course in case $\bigcap \mathrm{T} = id_L$, the identity relation, we obtain a Hausdorff T -tolerance topological lattice. By the way, for congruences some properties of congruence topologies on general algebras were considered in [3].

A map $[,]: L \times L \to L$ is called a commutator of L (see [5]) if for all $a_i \in L(i \in I)$, $a, b \in L$ the following hold:

$$\left[\bigvee_{i\in I} a_i, b\right] = \bigvee_{i\in I}[a_i, b];$$
$$[a,b] = [b,a];$$
$$[a,b] \le a \wedge b.$$

Our aim is to present a slight generalization of D.Schweigert's tolerances definable by commutators on lattices [5], then we connect this with tolerance topologies. Finally we give a necessary condition for strengthening of our tolerance to a congruence relation.

Definition. Let $[\,,\,]: L \times L \to L$ be a commutator of L. For $\bar{a} = (a_1, \dots, a_n) \in L^n$ we define a binary relation $\tau_{\bar{a}} \subseteq L \times L$ by following way: $(x, y) \in \tau_{\bar{a}}$ if and only if $[\dots[[x, a_1], a_2], \dots, a_n] \le y$ and $[\dots[[y, a_1], a_2], \dots, a_n] \le x$.

This definition generalize Schweigert's tolerance d_β [5] and we show that some properties of d_β are preserved by our tolerance $\tau_{\bar{a}}$.

Proposition 1. *$\tau_{\bar{a}}$ is a tolerance of L.*

Proof. Reflexivity and symmetry of $\tau_{\bar{a}}$ is obvious. Let $(x_1, y_1) \in \tau_{\bar{a}}$ and $(x_2, y_2) \in \tau_{\bar{a}}$. Then

$$[\dots[[x_1 \vee x_2, a_1], a_2], \dots, a_n] = [\dots[[x_1, a_1] \vee [x_2, a_1], a_2], \dots, a_n] = \dots =$$
$$= [\dots[[x_1, a_1], a_2], \dots, a_n] \vee [\dots[[x_2, a_1], a_2], \dots, a_n] \le y_1 \vee y_2$$

and similarly $[\ldots[[y_1 \vee y_2, a_1], a_2], \ldots, a_n] \le x_1 \vee x_2$. Hence $(x_1 \vee x_2, y_1 \vee y_2) \in \tau_{\bar a}$.

Obviously any commutator is monotone map. Consequently

$$[\ldots[[x_1 \wedge x_2, a_1], a_2], \ldots, a_n] \le [\ldots[[x_1, a_1], a_2], \ldots, a_n] \le y_1,$$

and similarly

$$[\ldots[[x_1 \wedge x_2, a_1], a_2], \ldots, a_n] \le y_2.$$

Hence

$$[\ldots[[x_1 \wedge x_2, a_1], a_2], \ldots, a_n] \le y_1 \wedge y_2.$$

By symmetry

$$[\ldots[[y_1 \wedge y_2, a_1], a_2], \ldots, a_n] \le x_1 \wedge x_2.$$

Hence $(x_1 \wedge x_2, y_1 \wedge y_2) \in \tau_{\bar a}$.

Thus the relation $\tau_{\bar a}$ is compatible with the lattice operations. □

Proposition 2. *For every* $\bar a \in L^n$ $[\ldots[[x, a_1], a_2], \ldots, a_n] = \Lambda\{y \in L | (x, y) \in \tau_{\bar a}\}$.

Proof. If $(x, y) \in \tau_{\bar a}$ then by definition of $\tau_{\bar a}$ we have $[\ldots[[x, a_1], a_2], \ldots, a_n] \le y$, hence $[\ldots[[x, a_1], a_2], \ldots, a_n] \le \Lambda\{y \in L | (x, y) \in \tau_{\bar a}\}$. On the other hand $[\ldots[[y, a_1], a_2], \ldots, a_n] \le x$ and so, by the monotony of a commutator,

$$[\ldots[[\ldots[[x, a_1], a_2], \ldots, a_n], a_1], \ldots, a_n] \le [\ldots[[y, a_1], a_2], \ldots, a_n] \le x.$$

Thus $(x, [\ldots[[x, a_1], a_2], \ldots, a_n]) \in \tau_{\bar a}$ and hence

$$[\ldots[[x, a_1], a_2], \ldots, a_n] \ge \Lambda\{y \in L | (x, y) \in \tau_{\bar a}\}.$$ □

Let $\bar 0$ (respectively, $\bar 1$) be the element $(0, \ldots, 0)$ (respectively, $(1, \ldots, 1)$) of L^n.

Proposition 3. *If* $\bar b \le \bar c$ *in* L^n *then* $\tau_{\bar b} \ge \tau_{\bar c}$*; in particular,* $\tau_{\bar 0} = L \times L$ *is the largest element and* $\tau_{\bar 1}$ *is the least element in the poset of all tolerances* $(\tau_{\bar a} | \bar a \in L)$.

Proof. If $(x, y) \in \tau_{\bar c}$ then $[\ldots[x, c_1], \ldots, c_n] \le y$ and $[\ldots[y, c_1], \ldots, c_n] \le x$. Hence $[\ldots[x, b_1], \ldots, b_n] \le [\ldots[x, c_1], \ldots, c_n] \le y$ and $[\ldots[y, b_1], \ldots, b_n] \le [\ldots[y, c_1], \ldots, c_n] \le x$, since the commutator $[\ ,\ ]$ is a monotone map. Thus, $(x, y) \in \tau_{\bar b}$. □

Proposition 4. *For every* $b, c \in L$ $\tau_{\bar b} \cap \tau_{\bar c} = \tau_{\bar b \vee \bar c}$, *where* $\bar b = (b, 1, \ldots, 1)$, $\bar c = (c, 1, \ldots, 1)$ *and* $\bar b \vee \bar c$ *the join in the lattice* L^n.

Proof. Obviously $\bar b \le \bar b \vee \bar c$ and $\bar c \le \bar b \vee \bar c$. So by Proposition 3 $\tau_{\bar b} \ge \tau_{\bar b \vee \bar c}$ and $\tau_{\bar c} \ge \tau_{\bar b \vee \bar c}$. Therefore $\tau_{\bar b} \cap \tau_{\bar c} \ge \tau_{\bar b \vee \bar c}$.

For the converse inclusion, if $(x, y) \in \tau_{\bar b}$ and $(x, y) \in \tau_{\bar c}$ then in particular

$$[\ldots[[x, b], 1], \ldots, 1] \le y \quad \text{and} \quad [\ldots[[x, c], 1], \ldots, 1] \le y.$$

Hence $[\ldots[[x, b \vee c], 1], \ldots] = [\ldots[[x, b], 1], \ldots, 1] \vee [\ldots[[x, c], 1], \ldots, 1] \le y$. By symmetry $[\ldots[[y, b \vee c], 1], \ldots, 1] \le x$. The last two inclusions imply that $(x, y) \in \tau_{\bar b \vee \bar c}$. □

Thus Propositions 1-3 and abovementioned notes imply the following

Theorem 1. *For any* $n \ge 1$ *the set* $\mathrm{T} = \{\tau_{\bar a} | \bar a = (a, 1, \ldots, 1) \in L^n\}$ *is a meet-subsemilattice of* $Tol(L)$, *the lattice of all tolerances of* L, *with the least element* $\tau_{\bar 1}$ *and the largest element* $\tau_{(0,1,1,\ldots,1)}$ *which is equal to* $\tau_{\bar 0} = L^2$. *Moreover,* $Tol(L)$ *is a* T*-tolerance topological lattice and if the commutator* $[,]$ *has the property* $[x, 1] = x$ *for every* $x \in L$ *then* $Tol(L)$ *is Hausdorff.*

For $\overline{a}=(a_1,...,a_n)\in L^n$ we say that $\overline{\overline{a}}=(a_1,...,a_n,a_1',...,a_n')$ is a continuation of $\overline{a}$ if $a_1\le a_1',...,a_n\le a_n'$ in L.

Theorem 2. *For any congruence* $\tau_{\overline{a}}$ *and any continuation* $\overline{\overline{a}}$ *of* $\overline{a}$

$$\tau_{\overline{a}}=\tau_{\overline{\overline{a}}}.$$

Proof. Obviously that $[...[[[...[[x,a_1],a_2],...,a_n],a_1'],a_2'],...,a_n']\le[...[x,a_1],...,a_n]$. Let $(x,y)\in\tau_{\overline{a}}$. Then $[...[x,a_1],...,a_n]\le y$ and hence $[...[[...[[x,a_1],a_2],...,a_n],a_1'],...,a_n']\le y$. Similarly $[...[[...[[y,a_1],a_2],...,a_n],a_1'],...,a_n']\le x$. Hence $(x,y)\in\tau_{\overline{\overline{a}}}$ and thus $\tau_{\overline{a}}\subseteq\tau_{\overline{\overline{a}}}$.

For the converse inclusion, let $(x,y)\in\tau_{\overline{\overline{a}}}$. H.-J.Bandelt [1] proved that in any lattice L a pair (x,y) belongs to tolerance ξ if and only if $(x\wedge y,x\vee y)$ belongs to ξ. So, without loss of generality we may assume that $x\le y$. Then $[...[[...[x,a_1],...,a_n],a_1'],...,a_n']\le y$ and $[...[[...[y,a_1],...,a_n],a_1'],...,a_n']\le x$. Obviously $[...[y,a_1],...,a_n]\le[...[y,a_1],...,a_n]$ and as the commutator $[,]$ is a monotone map, then

$$[...[[...[y,a_1],...,a_n],a_1],...,a_n]\le[...[[...[y,a_1],...,a_n],a_1'],...,a_n']\le x\le y.$$

Hence $(y,[...[y,a_1],...,a_n])\in\tau_{\overline{a}}$. This implies that

$$(y\vee x,[...[y,a_1],...,a_n]\vee x)=(y,[...[y,a_1],...,a_n]\vee x)\in\tau_{\overline{a}}.$$

On the other hand it is easy to see that

$$([...[y,a_1],...,a_n],[...[[...[y,a_1],...,a_n],a_1'],...,a_n'])\in\tau_{(a_1',...,a_n')}.$$

Consequently, as $(x,y)\in\tau_{\overline{\overline{a}}}$,

$$([...[y,a_1],...,a_n]\vee x,[...[[...[y,a_1],...,a_n],a_1'],...,a_n']\vee x)=$$
$$=([...[y,a_1],...,a_n]\vee x,x)\in\tau_{(a_1',...,a_n')}.$$

This and the predecessor paragraph imply that $(x,y)\in\tau_{\overline{a}}\circ\tau_{(a_1',...,a_n')}$.

By Proposition 3 $\tau_{(a_1',...,a_n')}\le\tau_{\overline{a}}$. Hence $(x,y)\in\tau_{\overline{a}}\circ\tau_{\overline{a}}=\tau_{\overline{a}}$; the last equality is true by the transitivity of the tolerance $\tau_{\overline{a}}$. Thus $(x,y)\in\tau_{\overline{a}}$ and therefore the inclusion $\tau_{\overline{a}}\supseteq\tau_{\overline{\overline{a}}}$ is proved. □

In the forthcoming paper we shall deal with a generalization of Theorem 2 and with applications to solvability and like properties of commutators of algebras.

References

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