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ON SOME PROPERTIES OF HOLOMORPHIC SOLUTIONS OF THE CLASS OF SECOND ORDER OPERATOR-DIFFERENTIAL EQUATIONS

Abstract

The Fragmen-Lindilef type theorem is proved for holomorph solutions of the second order operator-differential equation with normal operator in the main part in some sectors.

Let's consider the operator bunch in a separable Hilbert space H

$$P(\lambda)=-\lambda^2E+\lambda A_1+A_2+A^2 \tag{1}$$

and the connected operator-differential equation

$$P(d/dz)u(z)=-u''(z)+A_1u'(z)+A_2u(z)+A^2u(z),\quad z\in S_\alpha \tag{2}$$

where the derivatives are understood in the sense of complex analysis in the abstract Hilbert spaces:

$$S_\alpha=\{z:|\arg z|<\alpha\},\ 0<\alpha<\pi/2.$$

The operators A, A_1 and A_2 satisfy the following conditions:

1^0. A - is a normal operator, with quite continuous inverse A^{-1}, whose spectrum is contained on finite number of rays from the sector

$$S_\varepsilon=\{\lambda:|\arg\lambda|\le\varepsilon\},\ 0\le\varepsilon<\pi/2-\alpha$$

2^0. The operators $B_1=A_1A^{-1}, B_2=A_2A^{-2}$ are bounded in H, moreover the operator $E+B_2$ has a bounded inverse in H.

Let μ_n and e_n be the eigen-vectors and eigen-elements of the operator, i.e. $Ae_n=\mu_ne_n$, $\mu_n\in S_\varepsilon$. Let's denote by σ_p the sub-set of quite continuous operators B, for which

$$\sum_{n=1}^{\infty}\left(\mu_n\left(B^*B\right)^{1/2}\right)^p<\infty,$$

where μ_n is the n-th eigen-value of the operator B.

Let's define a class of functions $H_2(\alpha:H)$, consisting of functions $f(z)$ which are holomorphic in sector S_α, satisfying the condition:

$$\sup_{|\varphi|<\alpha}\left\|f\left(te^{i\varphi}\right)\right\|^2_{L_2(R_+:H)}\equiv\sup_{|\varphi|<\alpha}\int_0^\infty\left\|f\left(te^{i\varphi}\right)\right\|^2dt<\infty$$

As is known, the functions from the class $H_2(\alpha:H)$ have the boundary values $f\left(te^{\pm\alpha}\right)\in L_2(R_+:H)$ and integral of Cauchy type [1] holds:

$$f(z)=\frac{1}{2\pi i}\int_0^\infty\frac{f\left(te^{-i\alpha}\right)}{te^{-i\alpha}-z}e^{-i\alpha}dt-\frac{1}{2\pi i}\int_0^\infty\frac{f\left(te^{i\alpha}\right)}{te^{i\alpha}-z}e^{i\alpha}dt \tag{3}$$

Let's denote, that $H_2(\alpha:H)$ is a Hilbert space with respect to norm [1]

$$\|f\|_{2,\alpha}=\frac{1}{\sqrt{2}}\left(\left\|f\left(te^{i\alpha}\right)\right\|^2_{L_2(R_+:H)}+\left\|f\left(te^{-i\alpha}\right)\right\|^2_{L_2(R_+:H)}\right)^{1/2}$$

Further, let's denote by

$$W_2^2(\alpha:H)=\{u(z):u''(z)\in H_2(\alpha:H),\ A^2u(z)\in H_2(\alpha:H)\}$$

the hilbert space with norm

$$\|u\|_{2,\alpha}=\left(\|u''\|_{2,\alpha}^2+\|A^2u\|_{2,\alpha}^2\right)^{1/2}.$$

Let's denote, that for the vector-functions from $W_2^2(\alpha:H)$ holds the theorem on intermediate derivatives [2]:

$$\left\|A^{2-j}u^{(j)}\right\|_{2,\alpha}\le c\|u\|_{2,\alpha},\ j=0,1,2 \qquad (4)$$

Definition. *Let's call the functions $u(z)\in W_2^2(\alpha:H)$ as holomorph regular solution of equation (2), if the vector-function $u(z)$ satisfies equation (2) identically in S_α.*

In the present work we'll prove some confirmations on regular solutions of equation (2), in particular as Fragmen-Lindilef type theorem.

The analogical conformations in different situations are obtained in works, example [2,3,4].

It has place

Theorem 1. *The set L_0 of holomorph regular solutions is a close subset of the space $H_2(\alpha:H)$.*

Proof. The linearity of the set L_0 is evident. Let's prove the closeness of the set L_0. Let $u_n(z)\in W_2^2(\alpha:H)$ and $P_0u_n(z)=0$, when $z\in S_\alpha$. Suppose that $u_n(z)\to u(z)$ in the space $W_2^2(\alpha:H)$. Let's show that $u(z)\in L_0$. As $u_n^{(j)}(z)-u^{(j)}(z)\in W_2^2(\alpha:H)$, then on theorem about intermediate derivatives (4) has place the inequality:

$$\left\|A^{2-j}u_n^{(j)}(z)-A^{2-j}u^{(j)}(z)\right\|_{2,\alpha}\le C_j\|u_n-u\|_{2,\alpha},\quad j=0,1,2.$$

Consequently, $A^{2-j}u_n^{(j)}(z)\to A^{2-j}u^{(j)}(z)$ when $n\to\infty$ in space $W_2^2(\alpha:H)$, $j=0,1,2$. Let's show that this convergence is proportional in every compact $\mathcal{G}\subset S_\alpha$.

As $A^{2-j}u_n^{(j)}(z)\in H_2(\alpha:H)$, $A^{2-j}u^{(j)}(z)\in H_2(\alpha:H)$, then each of them has the bound any value $v_{n,j}(te^{\pm i\alpha})$ and $v_j(te^{\pm i\alpha})$ and on formula (3) holds the inequality:

$$\sup_{z\in\mathcal{G}}\left\|A^{2-j}u_n^{(j)}(z)-A^{2-j}u^{(j)}(z)\right\|\le\frac{1}{2\pi}\int_0^\infty\frac{\left\|v_{n,j}(te^{-i\alpha})-v_j(te^{-i\alpha})\right\|}{|te^{-i\alpha}-z|}dt+$$

$$+\frac{1}{2\pi}\int_0^\infty\frac{\left\|v_{n,j}(te^{i\alpha})-v_j(te^{i\alpha})\right\|}{|te^{i\alpha}-z|}dt\le$$

$$\le\frac{1}{2\pi}\left(\int_0^\infty\left\|v_{n,j}(te^{-i\alpha})-v_j(te^{-i\alpha})\right\|^2dt\right)^{1/2}\sup_{z\in\mathcal{G}}\left(\int_0^\infty\frac{1}{|te^{-i\alpha}-z|^2}dt\right)^{1/2}+$$

$$+\frac{1}{2\pi}\left(\int_0^\infty\left\|v_{n,j}(te^{i\alpha})-v_j(te^{i\alpha})\right\|^2dt\right)^{1/2}\sup_{z\in\mathcal{G}}\left(\int_0^\infty\frac{1}{|te^{i\alpha}-z|^2}dt\right)^{1/2}\le$$

$$\le const\left\|A^{n-j}u_n^{(j)}(z)-A^{n-j}u^{(j)}(z)\right\|_{2,\alpha}, \quad j=0,1,2.$$

In this way the sequence $A^{2-j}u_n^{(j)}(z)$ convergences to $A^{2-j}u^{(j)}(z)$ proportionally, in every compact $\mathcal{G}\subset S_\alpha$.

Further, from theorem about intermediate derivatives follows, that

$$\sup_{z\in\mathcal{G}}\left\|P(d/dz)u_n(z)-P(d/dz)u(z)\right\|\le$$

$$\le const\sum_{j=0}^{2}\sup_{z\in\mathcal{G}}\left\|A^{2-j}u_n^{(j)}(z)-A^{2-j}u^{(j)}(z)\right\|.$$

Taking into account that $P(d/dz)u_n(z)=0$ in the last inequality and taking the limit when $n\to\infty$, we get:

$$P(d/dz)u(z)=0, \text{ i.e. } u(z)\in L_0$$

The theorem is proved.

Let's denote by L_τ - the sub-set of holomorph regular solutions of equation (2), such that

$$L_\tau=\left\{u:u(z)\in L_0,\ e^{\tau z}u(z)\in H_2(\alpha:H)\right\},\ \tau\ge 0.$$

It holds the following

Theorem 2. *Let the conditions 1^0, 2^0 and one of the next conditions be fulfilled:*

1) $A^{-1}\in\sigma_p$ *when* $0<p\le\min\left(\dfrac{\pi}{\pi-2\alpha},\dfrac{\pi}{2\alpha}\right)$ *and it holds the inequality*

$$K(\varepsilon)=c_1(\varepsilon,\alpha)\|B_1\|+c_2(\varepsilon,\alpha)\|B_2\|<1 \tag{5}$$

where

$$c_1(\varepsilon,\alpha)=(2\cos(\alpha+\varepsilon))^{-1}$$

$$c_2(\varepsilon,\alpha)=\begin{cases}1, & npu\ \ 0<\alpha+\varepsilon\le\pi/4\\ \left(\sqrt{2}\cos(\alpha+\varepsilon)\right)^{-1}, & npu\ \ \pi/4\le\alpha+\varepsilon<\pi/2\end{cases} \tag{6}$$

2) $A^{-1}\in\sigma_p$ $(0<p<\infty)$, B_1,B_2 *are continuous operators in* H, $P^{-1}(\lambda)$ *exists on rays*

$$\Gamma_{\pm(\pi/2+\theta)}=\{z:\arg z=\pm(\pi/2+\theta)\}$$

on these rays it holds the proportional estimation

$$\left\|A^2P^{-1}(\lambda)\right\|+\left\|\lambda^2P^{-1}(\lambda)\right\|\le const. \tag{7}$$

Then, if $u(z)\in L_\tau$ *in all* $\tau\ge0$, *then* $u(z)\equiv0$.

Proof. If $u(z)\in L_0$, then its Laplace transformation $\hat{u}(\lambda)$ admit holomorph continuation in domain [7, p.221] $S_{\pi/2+\alpha}=\{\lambda:|\arg\lambda|<\pi/2+\alpha\}$.

Let's show that by fulfilling the condition (5), on rays $\Gamma_{\pm(\pi/2+\alpha)}$ and $\Gamma_{\mp(\pi/2-\alpha)}$ it holds estimation (7).

Indeed, let $\lambda\in\Gamma_{\pm(\pi/2+\alpha)}$ or $\lambda\in\Gamma_{\mp(\pi/2-\alpha)}$. Then $P_0(\lambda)=r^2e^{\pm2i\alpha}+A^2$ are inversable and from equality

$$P(\lambda)=P_0(\lambda)+P_1(\lambda)=\left(E+P_1(\lambda)P_0^{-1}(\lambda)\right)P_0(\lambda),\quad (P_1(\lambda)=\lambda A_1+A_2) \tag{8}$$

invertibility of follows, that for $P(\lambda)$ on rays $\Gamma_{\pm(\pi/2+\alpha)}$ and $\Gamma_{\mp(\pi/2-\alpha)}$ it is enough to prove, that on these rays $\left\|P_1(\lambda)P_0^{-1}(\lambda)\right\|<1$. Since on these rays

$$\left\|P_1(\lambda)P_0^{-1}(\lambda)\right\|\le\|B_1\|\left\|\lambda AP_0^{-1}(\lambda)\right\|+\|B_2\|\left\|A^2P_0^{-1}(\lambda)\right\|, \tag{9}$$

then let's estimate the norms $\|\lambda A P_0^{-1}(\lambda)\|$ and $\|A^2 P_0^{-1}(\lambda)\|$. From spectral decomposition of operator A we find, that

$$\|\lambda A P_0^{-1}(\lambda)\| = \sup_n \left| r\mu_n \left(r^2 e^{\pm 2i\alpha} + \mu_n^2\right)^{-1} \right| =$$

$$= \sup_n \left| r\mu_n \left[r^4 + |\mu_n|^4 + 2r^2 |\mu_n|^2 \cos 2(\alpha \mp \arg \mu_n)\right]^{-1/2} \right| \leq$$

$$\leq \sup_n \left| r|\mu_n| \left[r^4 + |\mu_n|^4 + 2r^2 |\mu_n|^2 \cos 2(\alpha + \varepsilon)\right]^{-1/2} \right| \leq$$

$$\leq \sup_n r|\mu_n| \left[2r^2 |\mu_n|^2 (1 + \cos 2(\alpha + \varepsilon))\right]^{-1/2} =$$

$$= (2\cos(\alpha + \varepsilon))^{-1} = c_1(\varepsilon).$$

By the same way

$$\|A^2 P_0^{-1}(\lambda)\| \leq \sup_n \left| \mu_n^2 \left[r^4 + |\mu_n|^4 + 2r^2 |\mu_n|^2 \cos 2(\alpha + \varepsilon)\right]^{-1/2} \right|.$$

When $0 < \alpha + \varepsilon \leq \pi/4$, $\cos 2(\alpha + \varepsilon) \geq 0$, therefore

$$\|A^2 P_0^{-1}(\lambda)\| \leq \sup_n |\mu_n|^2 \left(r^4 + |\mu_n|^4\right)^{-1/2} \leq 1,$$

but when $\pi/4 \leq \alpha + \varepsilon < \pi/2$, $\cos 2(\alpha + \varepsilon) \leq 0$, therefore

$$\|A^2 P_0^{-1}(\lambda)\| \leq \sup_n |\mu_n|^2 \left[\left(r^4 + |\mu_n|^4\right)(1 + \cos 2(\alpha + \varepsilon))\right]^{-1/2} =$$

$$= \left(\sqrt{2}\cos(\alpha + \varepsilon)\right)^{-1} = c_2(\varepsilon).$$

From obtained estimations and from equality (8) follows that on rays $\Gamma_{\pm(\pi/2+\alpha)}$ and $\Gamma_{\mp(\pi/2-\alpha)}$ estimation (7) holds.

Since $P^{-1}(\lambda)$ exists on rays $\Gamma_{\pm(\pi/2+\alpha)}$ and the inequality (7), holds then $u(z) \in L_0$ we can represent in the form

$$u(z) = \frac{1}{2\pi i} \int_{\Gamma_{(\pi/2+\alpha)}} \hat{u}(\lambda) e^{\lambda z} d\lambda - \frac{1}{2\pi i} \int_{\Gamma_{-(\pi/2-\alpha)}} \hat{u}(\lambda) e^{\lambda z} d\lambda,$$

where

$$\hat{u}(\lambda) = P^{-1}(\lambda)\left[(\lambda E - A_1)u(0) - u'(0)\right] \tag{10}$$

From condition $u(z) \in L_\tau$ for any $\tau \geq 0$ follows, that $\hat{u}(\lambda)$ holomorphically continues to domain

$$S_{\tau,\alpha} = \{\lambda : |\arg \lambda + \tau| < \pi/2 + \alpha\} \text{ for } \tau \geq 0$$

i.e. $\hat{u}(\lambda)$ eigen vector-function.

Since $P(\lambda) = (E + M(\lambda))(E + B_2)A^2$, where $M(\lambda) = \lambda T_1 + \lambda^2 T_2$, but $T_1 = B_1 A^{-1}(E + B_2)^{-1} \in \sigma_p$, $T_2 = B_2 A^{-2}(E + B_2)^{-1} \in \sigma_{p/2}$ then from lemma by Keldysh [5] it follows that $(E + M(\lambda))^{-1}$, consequently and $P^{-1}(\lambda)$ is represented in the form of relation of two eigen functions of order not higher than p and minimal type with order p. From the equality (10) follows, that $\hat{u}(\lambda)$ also has the order not higher than p and minimal type by order p.

By fulfilling condition 1) of the theorem on rays $\Gamma_{\pm(\pi/2+\alpha)}$ and $\Gamma_{\mp(\pi/2-\alpha)}$ it holds estimation (7) and the angle between neighbor rays equals to $\pi-2\alpha$ and 2α. Therefore when $0<p<\min(\pi/2\alpha, \pi/\pi-2\alpha)$ to vector-function $\hat{u}(\lambda)$ we can apply the Fragmen-Lindilef theorem ([8], p.211) and from estimation (7), and also from equality (10), we get that $\|u(\lambda)\|\le c(|\lambda|+1)^{-1}$, when $\lambda\in\mathbf{C}$.

So it is identically equal to zero, consequently $u(z)\equiv 0$. Let's prove the theorem by fulfilling condition (2).

Denote

$$S_\delta=\mathbf{C}\setminus\left(\bigcup_{i=1}^{S}\{\lambda:|\arg\lambda-\omega_i|<\delta\}\bigcup_{i=1}^{S}\{\lambda:|\arg\lambda-(\pi+\omega_i)|<\delta\}\right),$$

where a spectrum of the operator bunch $P_0(\lambda)=-\lambda^2E+A^2$ is contained on rays $\Gamma_{\omega_i}=\{\lambda:\arg\lambda=\omega_i\}$ and $\Gamma_{\pi+\omega_i}=\{\lambda:\arg\lambda=(\pi+\omega_i)\}$, and

$$0<\delta<\min\left(\pi/2p, \min_{i\ne j}|\omega_i-\omega_j|\right),\ i,j=1,\dots,m.$$

Then by Keldish's lemma [5,6], when $\lambda\overline{\in}S_\delta$

$$\lim_{|\lambda|\to\infty}\left\|P_1(\lambda)P_0^{-1}(\lambda)\right\|=0$$

therefore from equality (10) we have

$$\|\hat{u}(\lambda)\|\le c_1\left\|P^{-1}(\lambda)\right\|(|\lambda|+1)\le c_1\left\|P_0^{-1}(\lambda)\right\|\left(\left\|E+P_1(\lambda)P_0^{-1}(\lambda)\right\|^{-1}\right)(|\lambda|+1)\le$$

$$\le c_2(|\lambda|+1)^{-2}(|\lambda|+1)=c_2(|\lambda|+1)^{-1}.$$

Since the angle between the rays is less than π/p applying the theorem Fragmen-Lindilef again, we get, that $\hat{u}(\lambda)\equiv 0$, i.e. $u(z)\equiv 0$.

Theorem is proved.

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