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ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF THE BOUNDARY CROSSING TIMES

Abstract

The main result of this paper is three-term asymptotic expansion for distribution of boundary crossing times.

Introduction. Let $\xi_n, n \geq 1$ be independent identically distributed random variables (r.v.), determined on some probability space (Ω, F, P).

Let

$$S_0 = 0,\ S_n = \sum_{k=1}^{n} \xi_k,\ n \geq 1.$$

and

$$\tau_a = \inf\{n \geq 1 : S_n \geq f_a(n)\}$$

denote the first time that a random walk $S_n\ n \geq 0$ crosses a nonlinear boundary $f_a(t),\ a > 0,\ t > 0$.

In the theory of non-linear boundary problems for random walk more attention is given to studying of asymptotic expansion for probability $P(\tau_0 \leq n)$ when $a \to \infty (n = n(a) \to \infty)$.

The similar problems are studied in works [1,2] under various suppositions on boundaries $f_a(t)$ and distribution of r.v. ξ_1.

In work [1] for boundaries $f_a(t) = at^{\beta},\ 0 \leq \beta \leq 1$ first two-terms of asymptotic expansion for distribution τ_a are obtained.

The results of work [1] are generalized for sufficiently wide class of boundaries $f_a(t)$ in work [2].

In this paper work the results of work [2] are precised and the third term of asymptotic expansion of probability $P(\tau_a \leq n)$ is established.

2. Conditions and denotations.

We'll assume that $\rho_4 = E(\xi_1 - E\xi_1)^4 < \infty,\ v = E\xi_1 > 0$ and the boundary $f_a(t)$ satisfies the following conditions:

I) For any a the function $f_a(t)$ is increasing and continuously differentiable at $t > 0$ and $f_a(1) \uparrow \infty,\ a \to \infty$.

II) If $a \to \infty$ and $n = n(a) \to \infty$, so that

$$\frac{f_a(n)}{n} \to v \quad \text{and} \quad f_a(n) \to \theta \in [0, v)$$

III) For every a $\dfrac{f_a'(n)}{f_a'(n)} \to 1$ is satisfied when $\dfrac{m}{n} \to 1,\ n \to \infty$.

Denote by H a set of boundaries $f_a(t)$ satisfying conditions I), II), and III).

Give the following necessary denotations:

$$\chi_n = S_n - f_a(n), \quad c_n = \frac{f_a(n) - n\nu}{\sigma\sqrt{n}}, \quad \sigma^2 = D\xi_1,$$

$$R_a(n,x) = P(\tau_a \le n,\ \chi_n \le x),$$

$$T = \inf_{n\ge 1}(S - n\theta),\ I(x) = \int_{-\infty}^{x} P(T < y)dy,$$

$$S_n^* = \frac{S_n - n\nu}{\sigma\sqrt{n}}, \quad g(t) = Ee^{it\xi_1},\ t \in R.$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du\ , \quad \varphi(x) = \Phi'(x),\ x \in R.$$

The following polynomials connected with Ermit's polynomials [5] are used in further.

$$P_3(x) = \frac{\rho_3}{6\sigma^3}H_3(x),\ H_3(x) = x^3 - 3x,\ \rho_3 = E|\xi_1 - \nu|^3,$$

$$R_3(x) = -\frac{\rho_3}{6\sigma^3}H_2(x),\ H_2(x) = x^2 - 1,$$

$$R_4(x) = \frac{3\sigma^4 - \rho_4}{24\sigma^4}H_3(x) - \frac{\rho_3^3}{72\sigma^6}H_2(x),$$

here $H_2(x)$, $H_3(x)$ - Ermit polynomials.

Note that the general construction of polynomials $H_k(x), P_k(x)$ and $R_k(x)$ explicitly is explained in [5 p. 506-607].

3. Main result.

Theorem. *Let enumerated above conditions with respect to distributions of r.v* ξ_1 *and boundary* $f_a(t)$ *be satisfied. Moreover, suppose that for some* $m \ge 1$ *the function* $|g|^m$ *is integrable and* $c_n = O(1)$ *when* $a \to \infty$ *and* $n = n(a) \to \infty$. *Then at* $a \to \infty$

$$P(\tau_a \le n) = \Phi(-c_n) + \varphi(c_n)\left[\frac{\lambda}{\sqrt{n}} - \frac{R_3(c_n)}{\sqrt{n}} + \frac{\lambda}{n}P_3(c_n) - \frac{R_4(c_n)}{n}\right] + o(1/n),$$

where $\lambda = \dfrac{I(0)}{\sigma}$.

Remark 1. By reason of constants we will note the following. Let's denote $T^- = \max(0, -T)$ negative part of T. It is easy to see that

$$I(0) = \int_{-\infty}^{0} P(T < y)dy = E(T^-).$$

It is known that a characteristic function of r.v. T is given by the following formula [5]

$$\varphi(t) = Me^{itT} = \exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\int_{Z_n \le 0}\left(e^{itZ_n^-} - 1\right)dP\right\},$$

where

$$Z_n = S_n - n\theta,\ \ n \ge 1,\ \ EZ_1 = \nu - \theta = \delta > 0.$$

Then

$$E(T^-) = -i\psi(0) = \sum_{n=1}^{\infty}\frac{E(Z_n^-)}{n}$$

Consequently, for calculation of the constant $l(0)$ we get the formula

$$l(0)=\sum_{n=1}^{\infty}\frac{E\left(Z_n^-\right)}{n}.$$

Note that if ξ_1 has a normal distribution with mean $v>0$ and variance $0<\sigma^2<\infty$, then $Z_n=S_n-n\theta$ also have normal distribution with mean $n\delta$ and variance $n\sigma^2$. In this case one finds that [4]

$$l(0)=\sigma\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left\{\varphi\left(\frac{\delta\sqrt{n}}{\sigma}\right)-\frac{\delta\sqrt{n}}{\sigma}\Phi\left(-\frac{\delta\sqrt{n}}{\sigma}\right)\right\}.$$

4. Auxiliary facts.

The following lemmas are used in proof of Theorem.

Lemma 1. *Let $f_a(t)\in H$ and $0<E\xi_1<\infty$. Then in sense of convergence w.p. 1 when $a\to\infty$:*

1) $\tau_a\to\infty$

2) $\dfrac{f_a(\tau_a)}{\tau_a}\to v=E\xi_1$

3) $\dfrac{\tau_a}{n_a}\to 1$,

where $n_a=n_a(v)$ is the solution of the equation $f_a(n)=nv$, which exists for sufficiently large a [2].

The lemma is proved in [2]/

Lemma 2. *Suppose that the conditions of theorem 1 are satisfied. Then*

$$R_a(n,x)=\sigma^{-1}\varphi(c_n)l(x)\left[\frac{1}{\sqrt{n}}+\frac{P_3(c_n)}{n}\right]+o\left(\frac{1}{n}\right). \tag{1}$$

Remark 2. Note that in work [2] for cases of nonlattice random variables first form of expansion (1) was obtained.

The proof of lemma 2. Define

$$R_a(n,x,y)=P\left(\tau_a\le n,\ \chi_n\in(y,x]\right).$$

Let's divide an interval $(y,x]$ into equal parts and put

$$t_0=y,\ t_k y+\frac{k}{m}(x-y),\ t_k(t_{k-1},t_k],\ k=\overline{1,m}\ \text{ and }\ Q_a(n,k)=P\left(\tau_a\le n\right|_u\in l_k).$$

By the total probability formula we have

$$R_a(n,x,y)=\sum_{k=1}^{m}Q_a(n,k)P\left(\chi_n\in l_k\right). \tag{2}$$

By the definition of τ_a we have

$$\begin{aligned}Q_a(n,k)&=P\left(S_i>f_a(i),\ \exists i\in[1,n)\middle|\chi_n\in l_k\right)=\\&=P\left(S_n-S_i<S_n-f_a(i),\ \exists i\in[1,n)\middle|\chi_n\in l_k\right)=\\&=P\left(S_n-S_{n-i}<S_n-f_a(n-i),\ \exists i\in[1,n)\middle|\chi_n\in l_k\right),\end{aligned}$$

where $\Delta_{i,n}=f_a(n)-f_a(n-i),\ i=\overline{1,n}$.

It is easy to understand that

$$P\left(S_i<t_{k-1}+\Delta_{i,n},\exists i\in[1,n)\middle|\chi_n\in l_k\right)\le Q_a(n,k)\le P\left(S_i<t_k+\Delta_{i,n},\exists\in[1,n)\middle|\chi_n\in l_k\right). \tag{3}$$

Further we shall need the following lemma.

Lemma 3. *Suppose that r.v.* ξ_1 *has nonlattice distribution with* $E\xi_1 > 0$ *and* $D\xi_1 < \infty$ *and* $c_n = O(1)$ *when* $a \to \infty$, $n = n(a) \to \infty$.

Then for any $\varepsilon > 0$ *there exists an integer number* q_1 *such that for sufficiently large* a *and for* r, x *and* y *from bounded set*

$$\max_{k \le m} P(S_i \le r, \exists i \in [q_i, n) | \chi_n \in l_k) < \varepsilon . \tag{4}$$

Analogous statements of type (4) is proved in works [3,4]. The estimation (4) may be deduced from lemma 7 in [3] ([4]).

We continue the proof of lemma 2.

By of (4), from (3) we have

$$\begin{gathered} P(S_i < t_{k-1} + \Delta_{i,n}, \exists i \in [1,n) | \chi_n \in l_k) \le Q(n,k) \le \\ \le P(S_i < t_k + \Delta_{i,n}, \exists i \in [1,q_1) | \chi_n \in l_k) + P(S_i < t_k + \Delta_{i,n}, \exists i \in [q_1,n) | \chi_n \in l_k) \end{gathered} \tag{5}$$

Under the made assumptions with respect to boundaries $f_a(t)$ one finds that for any fixed i

$$\Delta_{i,n} \to i\theta \quad \text{for} \quad a \to \infty .$$

Then by virtue of estimation (4) for any $\varepsilon > 0$ and sufficiently large a we have

$$\max_{k \le m} P(S_i < t_k + \Delta_{i,n}, \exists i \in [q_1,n) | \chi_n \in l_k) \le \max_{k \le m} (S_i < t_k + \varepsilon, \exists i \in [q_1,n) | \chi_n \in l_k) \le \varepsilon , \tag{6}$$

where $S_i' = S_i - i\theta$, $ES_1' = \mu - \theta > 0$.

Now from (5) and (6) follows that

$$\begin{gathered} P(S_i < t_{k-1} + \Delta_{i,n}, \exists i \in [1,q_1) | \chi_n \in l_k) \le Q_a(n,k) \le \\ \le P(S_i < t_k + \Delta_{i,n}, \exists i \in [1,q_1) | \chi_n \in l_k) + \varepsilon \end{gathered} \tag{7}$$

Since $c_n = O(1)$ when $a \to \infty$, then from lemma 7 of work [3] follows that for any k

$$\lim_{a \to \infty} P(S_i < t_{k-1} + \Delta_{i,n}, \exists i \in [1,q_1) | \chi_n \in l_k) = P(S_i' < t_{k-1}, \exists i \in [1,q_1)) \tag{8}$$

and

$$\lim_{a \to \infty} P(S_i < t_k + \Delta_{i,n}, \exists \in [1,q_1) | \chi_n \in l_k) = P(S_i' < t_k, \exists i \in [1,q_1)) . \tag{9}$$

Now from (7), (8) and (9) we have

$$P(S_i' < t_{k-1}, \exists i \in [1,q1)) - \varepsilon \le Q_a(n,k) \le P(S_i' < t_k, \exists i \in [1,q_1)) + 2\varepsilon . \tag{10}$$

Further from strong law of large numbers follows that for any $\varepsilon > 0$ there exists a number q_2, such that $P(S_i' \le r, \exists i > q_2) < \varepsilon$ for all r from bounded set

$$P(S_i' \le r, \ \exists i \in [1,q_2)) < P(T \le r) \le P(S_i' \le r, \exists i \in [1,q_2)) < +\varepsilon . \tag{11}$$

Instead of q_1 and q_2, assuming $q = \max(q_1, q_2)$, from (10) and (11) we'll get that

$$P(T < t_{k-1}) - 2\varepsilon \le Q_a(n,k) \le P(T < t_k) + 2\varepsilon . \tag{12}$$

Substituting (12) in (2) we'll find that

$$\sum_{k=1}^{m} (P(T < t_{k-1}) - 2\varepsilon) P(\chi_n \in l_k) \le R_a(n,x,y) \le \sum_{k=1}^{m} (P(T < t_k) + 2\varepsilon) P(\chi_n \in l_k) . \tag{13}$$

At the made assumptions in the proved theorem there is Edgeworth expansion for density $f_n(x)$, $n \ge m$ of normalized sum S_n^* of the form ([5], p.602)

$$f_n(x) = \varphi(x) + \frac{1}{\sqrt{n}} P_3(x) \varphi(x) + o\left(1 \big/ \sqrt{n} \right), \tag{14}$$

when $n \to \infty$.

By the expansion (14) one finds that

$$P(\chi_n \in l_k) = P(S_n \in f_a(n) + l_k) = \frac{x-y}{m\sigma\sqrt{n}}\varphi(c_n) + \frac{x-y}{m\sigma n}P_3(c_n)\varphi(c_n)\varphi(c_n) + o\left(\frac{1}{n}\right). \quad (15)$$

By virtue of that $c_n = O(1)$, from (13) and (15) we have

$$\begin{aligned}-2\varepsilon + \frac{\varphi(c_n)}{\sigma\sqrt{n}}\sum_{k=1}^{m}\frac{x-y}{m}P(T < t_{k-1}) + \frac{P_3(c_n)}{\sigma n}\sum_{k=1}^{m}\frac{x-y}{m}P(T < t_{k-1}) + o\left(\frac{1}{n}\right) \le R_a(n,x,y) \le \\ \le 2\varepsilon + \frac{\varphi(c_n)}{\sigma\sqrt{n}}\sum_{k=1}^{m}\frac{x-y}{m}P(T < t_k) + \frac{\varphi(c_n)P_3(c_n)}{\sigma\sqrt{n}}\sum_{k=1}^{m}\frac{x-y}{m}P(T < t_k) + o\left(\frac{1}{n}\right).\end{aligned} \quad (16)$$

Choosing sufficiently large m and small ε, from (16) we have

$$R_a(n,x,y) = \frac{\varphi(c_n)}{\sigma\sqrt{n}}\int_y^x P(T < z)dz + \frac{\varphi(c_n)P_3(c_n)}{\sigma\sqrt{n}}\sum_{k=1}^{m}\frac{x-y}{m}P(T < t_k) + o\left(\frac{1}{n}\right).$$

Tending $y \to -\infty$, from the last relation we find statement of lemma 2.

Proof of Theorem . We have

$$P(\tau_0 \le n) = P(S_n \le f_a(n)) + P(\tau_0 \le n,\ S_n \le f_a(n)),$$
$$P(S_n \le f_a(n)) = 1 - F_n(c_n)$$

and

$$R_a(n,0) = P(\tau_a \le n,\ \chi_n \le 0) = P(\tau_a \le n,\ S_n \le f_a(n)).$$

Consequently,

$$P(\tau_a \le n) = 1 - F_n(c_n) + R_a(n,0). \quad (17)$$

By assumptions with respect to distributions of random variable ξ_1 in the proved theorem there is a Edgeworth expansion for distribution $F_n(x)$ of normalized sum S_n^* of the form ([5], p.604)

$$F_n(x) = \Phi(x) + \varphi(x)\left[\frac{R_3(x)}{\sqrt{n}} + \frac{R_4(x)}{n}\right] + o\left(\frac{1}{n}\right) \quad (18)$$

uniformly in $x \in R$.

From the lemma 2 when $x = 0$ follows that

$$R_a(n,0) = \frac{I(0)}{\sigma\sqrt{n}}\varphi(c_n) + \frac{I(0)}{\sigma n}\varphi(c_n)P_3(c_n) + o\left(\frac{1}{n}\right) \quad (19)$$

Substituting (18) and (19) in (17), we obtain the statement of Theorem.

Remark 3. Note that condition $c_n = O(1)$ when $n = n(a) \to \infty$, $a \to \infty$, equivalent to that

$$\frac{f_a(n)}{n} - v = O\left(\frac{1}{\sqrt{n}}\right),$$

when $n(a) \to \infty$ (see condition II).

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