

SADYKHOVA F.S.

**ON A SPECIAL REPRESENTATION BY SUMS OF FUNCTIONS OF LESS NUMBER OF VARIABLES**

**Abstract**

In the present paper it is established the criterion of a special representation of functions of many variables by sums of functions of less number of variables, where the coefficients of the representation have supplementary properties. This theorem takes the central place in construction of an extremal function in the corresponding best approximation problem.

**Formulation of the problem.**

Let the real function  $F(x)$ ,  $x = (x_1, \dots, x_n)$  be determined on the parallelepiped  $[a_1, b_1; \dots; a_n, b_n]$ .

Let's divide the set of variables  $\{x_1, \dots, x_n\}$  into the subsets

$$t_i = (x_{k_{i-1}+1}, x_{k_{i-1}+2}, \dots, x_{k_i}), \quad i = \overline{1, m}$$

$$0 = k_0 < k_1 < \dots < k_m = n.$$

We'll consider  $F(x)$  also as the function of  $m$  group of variables  $F(x) = \Phi(t)$ . The function  $\Phi(t) = \Phi(t_1, \dots, t_m)$  is determined in "parallelepiped"  $\Pi = [c_1, d_1; \dots; c_m, d_m]$ , where

$$c_i = (a_{k_{i-1}+1}, \dots, a_{k_i}), \quad d_i = (b_{k_{i-1}+1}, \dots, b_{k_i}).$$

Let's introduce the denotations:

$$A(\Phi, t_k) = \sum_{\substack{v_j=c_i, d_i \\ i=1, m, j \neq k}} \Phi(v_1, \dots, v_{k-1}, t_k, v_{k+1}, \dots, v_m).$$

Let  $\varphi_i(t_i)$ ,  $i = \overline{1, m}$  be real functions determined on "segments"  $[c_i, d_i]$ , and  $p_j$ ,  $j = \overline{1, 2^m}$  be all possible tops of the "parallelepiped".

The main results is:

**Theorem 1.** Let  $\sum_{j=1}^{2^m} \Phi(p_j) \neq 0$ . In order that the representation

$$\Phi(t) = \sum_{k=1}^m (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq m} \alpha_{i_1} \dots \alpha_{i_k} \frac{\varphi_1(t_{i_1}) \dots \varphi_m(t_{i_k})}{\varphi_1(t_{i_1}) \dots \varphi_k(t_{i_k})}, \quad (1)$$

where

$$\alpha_i = \frac{1}{2} [\varphi_i(c_i) + \varphi_i(d_i)] \quad (2)$$

hold it is necessary and sufficient that the function  $\Phi(t)$  satisfy the functional equation

$$\Phi(t) = \sum_{k=0}^m (-1)^{k-1} 2^{-k} \left\{ \sum_{j=1}^{2^m} \Phi(p_j) \right\}_{1 \leq i_1 < \dots < i_k \leq m}^{-(m-k-1)} \prod_{\substack{s=1, m \\ s \neq i_1, \dots, i_k}} A(\Phi, t_s). \quad (3)$$

**Proof. Necessity.** Let's denote

$$R_k(t) = (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq m} \alpha_{i_1} \dots \alpha_{i_k} \frac{\varphi_1(t_{i_1}) \dots \varphi_m(t_{i_k})}{\varphi_1(t_{i_1}) \dots \varphi_k(t_{i_k})}. \quad (4)$$

Then by (2)

$$\Phi(t) = \sum_{k=1}^m R_k(t). \quad (5)$$

From (4) we obtain

$$\begin{aligned} R_1(t) &= \sum_{i=1}^m \alpha_i \frac{\varphi_1(t_1) \dots \varphi_m(t_m)}{\varphi_i(t_i)} = \alpha_1 \varphi_2(t_2) \dots \varphi_m(t_m) + \\ &+ \alpha_2 \varphi_1(t_1) \varphi_3(t_3) \dots \varphi_m(t_m) + \dots + \alpha_m \varphi_1(t_1) \dots \varphi_m(t_m). \end{aligned}$$

Using the property (2) we'll have

$$\begin{aligned} R_1(c_1, t_2, \dots, t_m) + R_1(d_1, t_2, \dots, t_m) &= 2\alpha_1 \varphi_2(t_2) \dots \varphi_m(t_m) + \\ + \sum_{i=2}^m \alpha_i \frac{[\varphi_1(c_i) + \varphi_1(d_i)] \varphi_2(t_2) \dots \varphi_m(t_m)}{\varphi_i(t_i)} &= 2\alpha_1 \varphi_2(t_2) \dots \varphi_m(t_m) + \\ + 2\alpha_1 \sum_{i=2}^m \alpha_i \frac{\varphi_2(t_2) \dots \varphi_m(t_m)}{\varphi_i(t_i)} &\stackrel{\text{def}}{=} 2\alpha_1 \varphi_2(t_2) \dots \varphi_m(t_m) + S_{21}(t_2, \dots, t_m). \end{aligned} \quad (6)$$

Here it is used the denotation

$$S_{kl} = S_{kl}(t_2, \dots, t_m) = 2\alpha_1 \sum_{2 \leq i_2 < \dots < i_k \leq m} \alpha_{i_2} \dots \alpha_{i_k} \frac{\varphi_2(t_2) \dots \varphi_m(t_m)}{\varphi_{i_2}(t_{i_2}) \dots \varphi_{i_k}(t_{i_k})}, \quad (7)$$

where in the note  $S_{kl}$  the first index  $k$  points that in every term in (7) product of multipliers is involved and  $l$  points that one of the multipliers is  $\alpha_1$ .

Further

$$\begin{aligned} R_2(t) &= \sum_{1 \leq i_1 < i_2 \leq m} \alpha_{i_1} \alpha_{i_2} \frac{\varphi_1(t_1) \dots \varphi_m(t_m)}{\varphi_{i_1}(t_{i_1}) \dots \varphi_{i_2}(t_{i_2})} = \\ &= -\alpha_1 [\alpha_2 \varphi_3(t_3) \dots \varphi_m(t_m) + \dots + \alpha_m \varphi_2(t_2) \dots \varphi_{m-1}(t_{m-1})] - \alpha_2 [\alpha_3 \varphi_1(t_1) \varphi_4(t_4) \dots \varphi_m(t_m) + \\ &+ \alpha_4 \varphi_1(t_1) \varphi_3(t_3) \varphi_5(t_5) \dots \varphi_m(t_m) + \dots + \alpha_m \varphi_1(t_1) \dots \varphi_{m-1}(t_{m-1})] - \dots - \\ &- \alpha_{m-1} \alpha_m \varphi_1(t_1) \dots \varphi_{m-2}(t_{m-2}) = -\frac{1}{2} S_{21}(t_2, \dots, t_m) - \alpha_2 [\alpha_3 \varphi_1(t_1) \varphi_4(t_4) \dots \varphi_m(t_m) + \\ &+ \alpha_4 \varphi_1(t_1) \varphi_3(t_3) \varphi_5(t_5) \dots \varphi_m(t_m) + \dots + \alpha_m \varphi_1(t_1) \dots \varphi_{m-1}(t_{m-1})] - \dots - \\ &- \alpha_{m-1} \alpha_m \varphi_1(t_1) \varphi_2(t_2) \dots \varphi_{m-2}(t_{m-2}). \end{aligned} \quad (8)$$

From (8) by using property (2) and denotation (7) we obtain

$$\begin{aligned} R_2(c_1, t_2, \dots, t_m) + R_2(d_1, t_2, \dots, t_m) &= \\ &= -S_{21}(t_2, \dots, t_m) - 2\alpha_1 \sum_{2 \leq i_2 < i_3 \leq m} \alpha_{i_2} \alpha_{i_3} \frac{\varphi_2(t_2) \dots \varphi_m(t_m)}{\varphi_{i_2}(t_{i_2}) \varphi_{i_3}(t_{i_3})} = -S_{21}(t_2, \dots, t_m) - S_3(t_2, \dots, t_m). \end{aligned}$$

Fulfilling the above pointed operation over

$$\begin{aligned} R_3(t) &= \sum_{1 \leq i_1 < i_2 < i_3 \leq m} \alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \frac{\varphi_1(t_1) \dots \varphi_m(t_m)}{\varphi_{i_1}(t_{i_1}) \varphi_{i_2}(t_{i_2}) \varphi_{i_3}(t_{i_3})} = \\ &= \alpha_1 \sum_{2 \leq i_2 < i_3 \leq m} \alpha_{i_2} \alpha_{i_3} \frac{\varphi_2(t_2) \dots \varphi_m(t_m)}{\varphi_{i_2}(t_{i_2}) \varphi_{i_3}(t_{i_3})} + \sum_{1 \leq i_1 < i_2 < i_3 \leq m} \alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \frac{\varphi_1(t_1) \dots \varphi_m(t_m)}{\varphi_{i_1}(t_{i_1}) \varphi_{i_2}(t_{i_2}) \varphi_{i_3}(t_{i_3})}. \end{aligned}$$

We obtain

$$R_3(c_1, t_2, \dots, t_m) + R_3(d_1, t_2, \dots, t_m) = S_{31}(t_1, \dots, t_m) + S_{41}(t_1, \dots, t_m).$$

In general case we have

$$R_k(c_1, t_2, \dots, t_m) + R_k(d_1, t_2, \dots, t_m) = (-1)^{k-1} S_{k1}(t_2, \dots, t_m) + (-1)^{k-1} S_{(k+1)1}(t_2, \dots, t_m),$$

$$k = \overline{1, m}$$

where  $S_{(m+1)1}$  is absent.

The above expressed allows to write

$$\begin{aligned} \Phi(c_1, t_2, \dots, t_m) + \Phi(d_1, t_2, \dots, t_m) &= \sum_{k=1}^m [R_k(c_1, t_2, \dots, t_m) + R_k(d_1, t_2, \dots, t_m)] = \\ &= R_1(c_2, t_2, \dots, t_m) + R_1(d_1, t_2, \dots, t_m) + \sum_{k=2}^{m-1} [R_k(c_1, t_2, \dots, t_m) + R_k(d_1, t_2, \dots, t_m)] + \\ &+ [R_m(c_2, t_2, \dots, t_m) + R_m(d_1, t_2, \dots, t_m)] = 2\alpha_1\varphi_2(t_2)\dots\varphi_m(t_m) + S_{21}(t_2, \dots, t_m) + \\ &+ \sum_{k=2}^{m-1} (-1)^{k-1} [S_{k1}(t_2, \dots, t_m) + S_{(k+1)1}(t_2, \dots, t_m)] + (-1)^{m-1} S_{m1}(t_2, \dots, t_m) = 2\alpha_1\varphi_2(t_2)\dots\varphi_m(t_m). \end{aligned}$$

Thus, we obtain the formula

$$2\alpha_1\varphi_2(t_2)\dots\varphi_m(t_m) = \Phi(c_1, t_2, \dots, t_m) + \Phi(d_1, t_2, \dots, t_m). \quad (9)$$

Now fulfilling all above cited operations over  $k^*$  by a group of variables  $t_2, \dots, t_m$  we obtain

$$\begin{aligned} 2\alpha_2\varphi_1(t_1)\varphi_3(t_3)\dots\varphi_m(t_m) &= \Phi(t_1, c_2, t_3, \dots, t_m) + \Phi(t_1, d_2, t_3, \dots, t_m), \\ ..... \end{aligned} \quad (10)$$

$$2\alpha_m\varphi_1(t_1)\dots\varphi_m(t_m) = \Phi(t_1, \dots, t_{m-1}, c_m) + \Phi(t_1, \dots, t_{m-1}, d_m).$$

Taking into account the denotation

$$\Phi(t \setminus b_{i_1}, \dots, b_{i_k}) = \Phi(t_1, \dots, t_{i_1-1}, b_{i_1}, t_{i_1+1}, \dots, t_{i_k-1}, b_{i_k}, t_{i_k+1}, \dots, t_m)$$

and uniting correlations (9)-(10) we obtain

$$2\alpha_i \frac{\varphi_1(t_1)\dots\varphi_m(t_m)}{\varphi_i(t_i)} = \Phi(t \setminus c_i) + \Phi(t \setminus d_i), \quad i = \overline{1, m}. \quad (11)$$

We have made some transformations over every group of variables  $t_i$ ,  $i = \overline{1, m}$  of functions  $\Phi(t)$  of form (1) with the property (2) and have obtained the representation (11) with respect to the group of variables  $t_i$ ,  $i = \overline{1, m}$ . We obtain

$$4\alpha_i\alpha_j \frac{\varphi_1(t_1)\dots\varphi_m(t_m)}{\varphi_i(t_i)\varphi_j(t_j)} = \Phi(t \setminus (c_i, c_j)) + \Phi(t \setminus (c_i, d_j)) + \Phi(t \setminus (d_i, c_j)) + \Phi(t \setminus (d_i, d_j)), \quad (12)$$

$$i, j = \overline{1, m}, \quad i \neq j, 1 \leq i < j \leq m.$$

Operating similarly over (12) we'll have

$$\begin{aligned} 8\alpha_i\alpha_j\alpha_k \frac{\varphi_1(t_1)\dots\varphi_m(t_m)}{\varphi_i(t_i)\varphi_j(t_j)\varphi_k(t_k)} &= \Phi(t \setminus (c_i, c_j, c_k)) + \\ &+ \Phi(t \setminus (c_i, c_j, d_k)) + \Phi(t \setminus (c_i, d_j, c_k)) + \Phi(t \setminus (d_i, c_j, c_k)) + \\ &+ \Phi(t \setminus (d_i, c_j, d_k)) + \Phi(t \setminus (d_i, d_j, c_k)) + \Phi(t \setminus (c_i, d_j, d_k)) + \Phi(t \setminus (d_i, d_j, d_k)). \end{aligned} \quad (13)$$

Going on with this process for any  $k = \overline{1, m}$  we'll have

$$2^k \alpha_{i_1} \dots \alpha_{i_k} \frac{\varphi_1(t_1)\dots\varphi_m(t_m)}{\varphi_{i_1}(t_{i_1})\dots\varphi_{i_k}(t_{i_k})} = \sum_{\substack{b_{j_l} = c_{i_l}, d_{i_l} \\ b_{j_l} = c_{i_{l+1}}, d_{i_{l+1}} \\ \dots \\ b_{j_k} = c_{i_k}, d_{i_k}}} \sum_{\substack{j=1, k \\ 1 \leq i_1 < \dots < i_k \leq m}} \Phi(t \setminus (b_{i_1}, \dots, b_{i_k})). \quad (14)$$

For  $k = m-1$  and  $k = m$  we'll obtain

$$2^{m-1} \frac{\alpha_1 \dots \alpha_m}{\alpha_i} \varphi_i(t_i) = A(\Phi, t_i), \quad i = \overline{1, m}, \quad (15)$$

$$2^m \alpha_1 \dots \alpha_m = \sum_{j=1}^{2^m} \Phi(P_j). \quad (16)$$

Let's determine the function  $\varphi_i(t_i)$  from (15)

$$\varphi_i(t_i) = 2^{-(m-1)} \frac{\alpha_i}{\alpha_1 \dots \alpha_m} A(\Phi, t_i).$$

Using the representation  $\varphi_i(t_i)$  we have

$$\begin{aligned} \alpha_1 \dots \alpha_k \frac{\varphi_1(t_1) \dots \varphi_m(t_m)}{\varphi_i(t_i) \dots \varphi_k(t_k)} &= \alpha_1 \dots \alpha_k \frac{2^{-m(m-1)} A(\Phi, t_1) \dots A(\Phi, t_m)}{2^{-k(m-1)} A(\Phi, t_1) \dots A(\Phi, t_k)} \times \\ &\times \frac{\alpha_1 \dots \alpha_m}{(\alpha_1 \dots \alpha_m)^m} \times \frac{(\alpha_1 \dots \alpha_m)^k}{\alpha_1 \dots \alpha_k} = 2^{(-m+k)(m-1)} (\alpha_1 \dots \alpha_m) \frac{A(\Phi, t_1) \dots A(\Phi, t_m)}{A(\Phi, t_1) \dots A(\Phi, t_k)}. \end{aligned} \quad (17)$$

The last fraction can be written in the form

$$\frac{A(\Phi, t_1) \dots A(\Phi, t_m)}{A(\Phi, t_1) \dots A(\Phi, t_k)} = \prod_{\substack{S=1, m \\ S \neq i_1, \dots, i_k}} A(\Phi, t_s). \quad (18)$$

Moreover from (16)

$$\alpha_1 \dots \alpha_m = 2^{-m} \sum_{j=1}^{2^m} \Phi(P_j). \quad (19)$$

Using (18) and (19) in (17) we obtain

$$\begin{aligned} \alpha_1 \dots \alpha_k \frac{\varphi_1(t_1) \dots \varphi_m(t_m)}{\varphi_i(t_i) \dots \varphi_k(t_k)} &= 2^{(-m+k)(m-2)} 2^{-m[-(m-k-1)]} \left[ \sum_{j=1}^{2^m} \Phi(P_j) \right]^{-(m-k-1)} \times \\ &\times \prod_{\substack{S=1, m \\ S \neq i, m, i_k}} A(\Phi, t_s) = 2^{-k} \left\{ \sum_{j=1}^{2^m} \Phi(P_j) \right\}^{-(m-k-1)} \prod_{S=1, m} A(\Phi, t_s). \end{aligned} \quad (20)$$

Now we must substitute the obtained expression in (20) to the expression of  $\Phi(t)$  from (1), that leads to the desired representation (3)

$$\Phi(t) = \sum_{k=1}^m (-1)^{k-1} 2^{-k} \left\{ \sum_{j=1}^{2^m} \Phi(P_j) \right\}^{-(m-k-1)} \sum_{1 \leq i_1 < \dots < i_k \leq m} \prod_{s=1, m} A(\Phi, t_s). \quad (3)$$

The necessity has been established.

**Sufficiency.** Choose  $\alpha_i, i = \overline{1, m}$  subjected to the condition

$$\alpha_1 \dots \alpha_m = 2^{-m} \sum_{j=1}^{2^m} \Phi(P_j). \quad (21)$$

Since by the statement of the theorem  $\sum_{j=1}^{2^m} \Phi(P_j) \neq 0$ , then from (22) we obtain

$$\alpha_i \neq 0, i = \overline{1, m}.$$

Let's determine the function  $\varphi_i(t_i)$  from

$$\varphi_i(t_i) = 2^{-(m-1)} \frac{\alpha_i}{\alpha_1 \dots \alpha_m} A(\Phi, t_i). \quad (22)$$

Using (21) the functional equation (3) can be written in the form

$$\Phi(t) = \sum_{k=1}^m (-1)^{k-1} 2^{-k} [2^m \alpha_1 \dots \alpha_m]^{-(m-k-1)} \sum_{1 \leq i_1 < \dots < i_k \leq m} \prod_{S=1, m, S \neq i_1, \dots, i_k} A(\Phi, t_s). \quad (23)$$

Using the expression of  $A(\Phi, t_i)$  from (22) we have:

$$\begin{aligned} \prod_{S=1, m, S \neq i_1, \dots, i_k} A(\Phi, t_s) &= \prod_{\substack{S=1, m \\ S \neq i_1, \dots, i_k}} 2^{m-1} \varphi_s(t_s) \frac{\alpha_1 \dots \alpha_m}{\alpha_s} = 2^{(m-1)(m-k)} \frac{\varphi_1(t_1) \dots \varphi_m(t_m)}{\varphi_{i_1}(t_{i_1}) \dots \varphi_{i_k}(t_{i_k})} \frac{(\alpha_1 \dots \alpha_k)^{m-k}}{\alpha_{i_1} \dots \alpha_{i_k}} = \\ &= 2^{(m-1)(m-k)} \alpha_{i_1} \dots \alpha_{i_k} (\alpha_1 \dots \alpha_m)^{(m-k-1)} \frac{\varphi_1(t_1) \dots \varphi_m(t_m)}{\varphi_{i_1}(t_{i_1}) \dots \varphi_{i_k}(t_{i_k})}. \end{aligned} \quad (24)$$

Using (24) in (23) we obtain

$$\begin{aligned} \Phi(T) &= \sum_{k=1}^m (-1)^{k-1} 2^{-k} \cdot 2^{-m(m-k-1)} \cdot 2^{(m-1)(m-k)} \times \\ &\times (\alpha_1 \dots \alpha_m)^{-m(m-k-1)} \cdot (\alpha_1 \dots \alpha_m)^{m-k-1} \sum_{1 \leq i_1 < \dots < i_k \leq m} \alpha_{i_1} \dots \alpha_{i_k} \frac{\varphi_1(t_1) \dots \varphi_m(t_m)}{\varphi_{i_1}(t_{i_1}) \dots \varphi_{i_k}(t_{i_k})} = \\ &= \sum_{k=1}^m (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq m} \alpha_{i_1} \dots \alpha_{i_k} \frac{\varphi_1(t_1) \dots \varphi_m(t_m)}{\varphi_{i_1}(t_{i_1}) \dots \varphi_{i_k}(t_{i_k})}. \end{aligned}$$

We have obtained the required representation of the function  $\Phi(t)$  from (1).

Now we must show that thus determinations  $\alpha_i$ ,  $i = \overline{1, m}$  satisfy the correlation (2).

From (21) we have

$$2\alpha_i = 2^{-(m-1)} \frac{\alpha_i}{\alpha_1 \dots \alpha_m} \sum_{j=1}^{2^m} \Phi(P_j). \quad (25)$$

Further, the expression  $A(\Phi, t_i)$  admits to write

$$A(\Phi, c_i) + A(\Phi, d_i) = \sum_{j=1}^{2^m} \Phi(P_j), \quad i = \overline{1, m}.$$

Using it in (25) we obtain

$$2\alpha_i = 2^{-(m-1)} \frac{\alpha_i}{\alpha_1 \dots \alpha_m} [A(\Phi, c_i) + A(\Phi, d_i)],$$

which by virtue of determination  $\varphi_i(t_i)$  from (22), taking  $k = y$

$$2\alpha_i = \varphi_i(c_i) + \varphi_i(d_i),$$

and that is (2).

The sufficiency has been proved. The theorem has been completely proved.

**Remark.** As it is seen from the proof the constants  $\alpha_i$ ,  $i = \overline{1, m}$  in expansion (1) of the function  $\Phi(t)$  can be chosen only arbitrary satisfying only (2).

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#### References

- [1]. Бабаев М.-Б.А. Прямые теоремы для приближения функций многих переменных суммами функций меньшего числа переменных. Специальные вопросы теории функций, т.2, Баку, 1980, с.3-40.

[2]. Бабаев М.-Б.А. *Об одной обратной задаче теории приближения*. Конструктивная теория функций, 77, София, 1980, с.9-15.

**Sadykhova F.S.**

Institute of Mathematics & Mechanics AS of Azerbaijan Republic,  
9, F. Agayev str., 370141, Baku, Azerbaijan.  
Tel.: 39-47-20 (off.), 61-25-25 (apt.).

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