Proceedings of IMM of Azerbaijan AS 121
2000 VOL. XIII(XXI)

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THE INVERSE PROBLEM ON DEFINITION OF COEFFICIENTS FOR DIFFERENTIAL EQUATIONS OF STATIONARY THERMO-ELASTICITY

Abstract

The article deals with problems on finding coefficients of differential equations which characterizes the model of stationary thermo-elasticity in non-homogeneous orthotropic material.

Mathematical simulation of thermo-elasticity processes is related with solving inverse problems on definition of thermo-elastic characteristics of materials. Definition of thermo-elastic characteristics is led to the problem on determination of coefficients of differential equations that are contained in mathematical model of the considered thermo-elastic process.

The problems on definition of thermo-elastic characteristics of inhomogeneous orthotropic material are considered in the paper.

Consider a process described by the following system of stationary thermo-elasticity equations

$$\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left[a_{ij}(x)\left(\frac{\partial u_i}{\partial x_j}+\varepsilon_{ij}\frac{\partial u_j}{\partial x_i}\right)\right]+\frac{\partial}{\partial x_i}\sum_{j=1}^{3}\varepsilon_{ij}b_{ij}(x)\frac{\partial u_j}{\partial x_j}-$$

$$-\frac{\partial}{\partial x_i}\left[\beta(x)T(x,y)\right]+F_i(x,y)=0\ ,\quad \left(i=\overline{1,3}\right), \tag{1}$$

$$-\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left[\lambda_j(x)\frac{\partial T}{\partial x_j}\right]+\alpha(x)T(x,y)=F_4(x,y), \tag{2}$$

where $\varepsilon_{ij}=1-\delta_{ij}$, δ_{ij} is a Kronecker symbol, $u_i\equiv u_i(x,y)$ are components of a displacement vector, $T\equiv T(x,y)$ is a temperature distribution in material, $a_{ij}(x)$, $b_{ij}(x)$, $\lambda_j(x),\alpha(x)$, $\beta(x)$ are thermo-elastic characteristics of material that do not depend on $y, F_k(x,y)$, $k=\overline{1,4}$ are given continuous functions in the domain

$$D=\Pi\times[0,1],\quad x\equiv(x_1,x_2)\in\Pi\ ,\quad y\in[0,1].$$

We give for the system (1) the following boundary conditions

$$u_i\big|_{y=0}=u_i\big|_{y=1}=0\ ,\quad T\big|_{y=0}=T\big|_{y=1}=0\ , \tag{3}$$

$$u_i\big|_{\Gamma\times[0,1]}=f_i(\xi,y)\ ,\quad T\big|_{\Gamma\times[0,1]}=f_4(\xi,y)\ ,\quad i=\overline{1,3}\ , \tag{4}$$

where $f_k(\xi,y)$, $k=\overline{1,4}$ are given continuous functions.

If the coefficients of the system (1), (2) and boundary functions from (4) are given, the system (1)-(4) makes possible to determine the components of a displacement vector $u_i(x,y)$ and a temperature field $T(x,y)$.

We are interested in solution of the inverse problem on definition of coefficients of the system (1), (2).

To this end we join to the system one or some of the following conditions depending on how many of coefficients of the equation are unknown:

$$u_i(x,\eta_i)=\gamma_i(x)\ ,\quad T(x,\eta_4)=\gamma_4(x)\ ,\quad \eta_k\in(0,1)\ , \tag{5}$$

$$\left.\frac{\partial u_i}{\partial y}\right|_{y=\eta_i} = g_i(x)\left(i=\overline{1,3}\right),\ \left.\frac{\partial T}{\partial y}\right|_{y=\eta_4} = g_4(x),\ \ \eta_k \in (0,1), \tag{6}$$

$$a_{3i}(x)\left.\frac{\partial u_i}{\partial y}\right|_{y=\eta_i} = g_i(x)\ i=\overline{1,3},\ \lambda_3(x)\left.\frac{\partial T}{\partial y}\right|_{y=\eta_4} = g_4(x), \tag{7}$$

where $\gamma_k(x), g_k(x), q_k(x), k=\overline{1,4}$ are the given continuous functions satisfying the necessary agreement conditions, $\eta_k, k=\overline{1,4}$ are the given numbers. In particular the equality $\eta_k = \eta_j, k \neq j$ is assumed. Each of these relations has a real physical meaning. They axpress critics of the practice to which a mathematical model of the studied process must satisfy.

The problem (1)-(6) under given coefficients and the right hand sides is a boundary value problem for a system of thermo-elasticity equations [1]. Assume that this problem has a unique solution. Consider a problem on definition of a part coefficients of the system (1), (2).

Let's $\omega_1(y)=\sqrt{2}\sin\sqrt{\mu}\,y,\ \omega_2(y)=\sqrt{2}\sin\sqrt{\nu}\,y,\ \mu=n^2\pi^2,\ \nu=m^2\pi^2,\ m, n$ are natural numbers and the condition A is fulfilled: $F_i(x,y)=F_{0i}(x)\omega_1(y)$, $F_4(x,y)=F_{04}(x)\omega_2(y)$, $f_i(\xi,y)=f_{0i}(\xi)\omega_1(y)$, $f_4(\xi,y)=f_{04}(\xi)\omega_2(y)$, F_{0k}, f_{0k} are given continuous functions.

The coefficients of higher derivatives in the system (1), (2) are sought in a class of positive, continuous and finite functions in the domain Π, and the remaining coefficients are only continuous and finite in this domain.

Multiply each equation of the system (1) by $\omega_1(y)$ and the equation (2) by $\omega_2(y)$ and integrate the obtained expressions on domain $(0,1)$. If we denote

$$\vartheta_i(x)=\int_0^1 u_i(x,y)\omega_1(y)dy,\ T_j(x)=\int_0^1 T(x,y)\omega_j(y)dy \tag{8}$$

we get

$$\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left[a_{ij}(x)\left(\frac{\partial \vartheta_i}{\partial x_j}+\varepsilon_{ij}\frac{\partial \vartheta_j}{\partial x_i}\right)\right]+\frac{\partial}{\partial x_i}\sum_{j=1}^{2}\varepsilon_{ij}b_{ij}(x)\frac{\partial \vartheta_j}{\partial x_j}-\mu a_{i3}(x)\vartheta_i(x)-$$

$$-\frac{\partial}{\partial x_i}\left[\beta(x),T_1(x)\right]+F_{0i}(x)=0,\ \ i=1,2, \tag{9}$$

$$\sum_{j=1}^{2}\frac{\partial}{\partial x_i}\left[a_{3j}(x)\frac{\partial \vartheta_3}{\partial x_j}\right]-\mu\, a_{33}(x)\vartheta_3(x)+F_{03}(x)=0, \tag{10}$$

$$-\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left[\lambda_j(x)\frac{\partial T_2}{\partial x_j}\right]-\left[\nu\,\lambda_3(x)+\alpha(x)\right]T_2(x)=F_{04}. \tag{11}$$

After a change of (8) the conditions (4)-(7) will adopt the form:

$$\vartheta_i(x)\big|_\Gamma = f_{0i}(\xi),\quad T_2(x)\big|_\Gamma = f_{04}(x), \tag{12}$$

$$\vartheta_i(x)\omega_1(\eta_i)=\gamma_i(x),\ \ T_2(x)\omega_2(\eta_4)=\gamma_4(x), \tag{13}$$

$$\vartheta_i(x)\omega_{1y}(\eta_i)=g_i(x),\ \ T_2(x)\omega_{2y}(\eta_4)=g_4(x), \tag{14}$$

$$a_{ii}(x)\vartheta_i(x)\omega_{1y}(\eta_i)=q_1(x),\ \lambda_3(x)T_2(x)\omega_{2y}(\eta_4)=q_4(x). \tag{15}$$

Consequently, under the assumption A, the system (1)-(7) equivalently will be transformed to the system (9)-(14).

Therefore, below we shall consider mainly the system (9)-(14).

Let first of all consider a problem on determination of coefficients for higher derivatives, of the system (1)-(2). Assume $a_{ij}(x), i \neq j$, $b_{12}(x)$, $\beta_i(x), \alpha(x)$ are continuous functions given on Π and from conditions (1)-(4) we must define $a_{ii}(x)$. Considering (13) in (9)-(11), we get:

$$\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left\{a_{ij}(x)\left[\frac{\partial\gamma_i}{\partial x_j}+\varepsilon_{ij}\frac{\omega_1(\eta_i)}{\omega_1(\eta_j)}\frac{\partial\gamma_j}{\partial x_i}\right]\right\}+\frac{\partial}{\partial x_i}\sum_{j=1}^{2}\varepsilon_{ij}b_{ij}(x)\frac{\omega_1(\eta_i)}{\omega_1(\eta_j)}\frac{\partial\gamma_j}{\partial x_j}-\mu a_{i3}(x)\gamma_i(x)-$$
$$-\frac{\omega_1(\eta_i)}{\omega_1(\eta_4)}\frac{\partial\gamma_j}{\partial x_j}\beta\gamma_4+\omega_1(\eta_i)F_{0i}(x)=0, \tag{16}$$

$$\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left[a_{3j}(x)\frac{\partial\gamma_3}{\partial x_j}\right]-\mu a_{33}(x)\gamma_3(x)+\omega_1(\eta_3)F_{03}(x)=0,$$

$$-\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left[\lambda_j(x)\frac{\partial\gamma_4}{\partial x_j}\right]+v\gamma_4(x)\lambda_3(x)=F_{04}(x)\omega_2(\eta_4)-\alpha\gamma_4(x)=0. \tag{17}$$

Hence, we define

$$a_{33}(x)=\frac{1}{\mu\gamma_3}\left\{\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left[a_{3j}(x)\gamma_3(x)\right]+\omega_1(\eta_3)F_{03}(x)\right\}.$$

A $a_{ii}(x)$ are defined from the system of the following partial differential equations of the first order

$$\frac{\partial a_{ii}}{\partial x_i}\frac{\partial\gamma_i}{\partial x_i}+a_{ii}(x)\frac{\partial^2\gamma_i}{\partial x_i^2}=-\Phi_i(x),\quad i=1,2, \tag{18}$$

where

$$\Phi_i(x)=\sum_{j=1}^{2}\left\{\frac{\omega_1(\eta_i)}{\omega_1(\eta_j)}\frac{\partial}{\partial x_i}\left[b_{ij}(x)\frac{\partial\gamma_j}{\partial x_j}\right]+\frac{\partial}{\partial x_j}\left[a_{ij}(x)\left(\frac{\partial\gamma_i}{\partial x_j}+\frac{\omega_1(\eta_i)}{\omega_1(\eta_j)}\frac{\partial\gamma_j}{\partial x_i}\right)\right]\right\}-$$
$$-\mu a_{i3}(x)\gamma_i(x)-\frac{\omega_1(\eta_i)}{\omega_1(\eta_4)}\frac{\partial}{\partial x_i}(\beta_i\gamma_4)+\omega_1(\eta_i)F_{0i}(x).$$

Let the functions $\dfrac{\partial\gamma_i}{\partial x_i}$, $i=1,2$ vanish only at points $x_i=x_i^0$, where x_i^0 are the numbers from belonging domain x_i (this condition may be substituted by the condition of the task at a point of a definition domain of the function itself). Then we obtain from (18)

$$a_{ii}(x)\Big|_{x_i=x_i^0}=\left[\left(\frac{\partial^2\gamma_i}{\partial x_i^2}\right)^{-1}\Phi_i(x)\right]_{x_i=x_i^0},\quad i=1,2. \tag{19}$$

Solving equation (18) under the condition (19), we get

$$a_{ii}(x)=\exp\left[-\int_{x_i^0}^{x_i}\left(\frac{\partial\gamma_i}{\partial x_i}\right)^{-1}\frac{\partial^2\gamma_i}{\partial x_i^2}dx_i\right]\left\{-\int_{x_i^0}^{x_i}\left(\frac{\partial\gamma_i}{\partial x_i}\right)^{-1}\Phi_i(x)\exp\left[\int_{x_i^0}^{x_i}\left(\frac{\partial\gamma_i}{\partial x_i}\right)^{-1}\frac{\partial^2\gamma_i}{\partial x_i^2}dx_i\right]dx_i-\right.$$
$$\left.-\left[\left(\frac{\partial^2\gamma_i}{\partial x_i^2}\right)^{-1}\Phi_i(x)\right]_{x_i=x_i^0}\right\}. \tag{20}$$

And one of functions $\lambda_j(x),\ j=\overline{1,3}$ is defined from the equation (17). Note that this equation doesn't consider the heatconductivity anisotropy in a general form. However, it consideres a thermal anisotropy of those solids, whose crystal structures belong to rhombic, quadrangular, trigonal, hexagonal, cubic system [3].

For all these systems, it holds the formula:

$$\lambda_3(x)=\frac{1}{\mu\gamma_4}\left[\omega_2(\eta_4)F_{04}(x)-\alpha\gamma_4+\sum_{j=1}^{2}\frac{\partial}{\partial x_j}(\lambda_j)\frac{\partial\gamma_4}{\partial x_j}\right]. \qquad (21)$$

If for a rhombic system $\lambda_3(x)$ and $\lambda_1(x)$ (or $\lambda_2(x)$) are known for the definition of $\lambda_2(x)$ (correspondingly $\lambda_1(x)$) we obtain a partial differential equation of the first order from which under the condition $\lambda_2(x_1,x_2^0)=f_2(x_1)$ we determine:

$$\lambda_2(x)=\left(\frac{\partial\gamma_4}{\partial x_2}\right)^{-1}\left\{f_2(x_1)\left(\frac{\partial\gamma_4}{\partial x_2}\right)_{x_2=x_2^0}+\int_{x_2^0}^{x_2}\left[\mu\gamma_4(x)\lambda_3(x)+\alpha\gamma_4(x)-\right.\right.$$

$$\left.\left.-\,\omega_2(\eta_4)F_{04}(x)-\frac{\partial}{\partial x_1}\left(\lambda_1\frac{\partial\gamma_4}{\partial x_1}\right)\right]dx_2\right\}, \qquad (22)$$

where it is assumed that $x_2^0\in\Pi$ is a given point, $f(x_1)$ is a given continuous function, and $\left(\frac{\partial\gamma_4}{\partial x_2}\right)_{x_2=x_2^0}\neq 0$. If for quadrangular, trigonal and hexagonal systems the function $\lambda_3(x)$ is known, the definition $\lambda_j(x)$ is led to the solution of the first order differential equation. A general integral of this equation makes possible to determine $\lambda_j(x),\ j=1,2$.

When a considered solid is of a crystal structure of a cubic system, the definition of a heatconductivity coefficient $\lambda(x)$ is reduced to the solution of the equation

$$\frac{\partial\gamma_4}{\partial x_1}\frac{\partial\lambda}{\partial x_1}+\frac{\partial\gamma_4}{\partial x_2}\frac{\partial\lambda}{\partial x_2}=(\nu\gamma_4-\Delta\gamma_4)\lambda+\alpha\gamma_4-\omega_2(\eta_4)F_{04}(x). \qquad (23)$$

By observing the demands of the theorem on the existence of an implicit function, we define from the equality

$$\Phi\left(\lambda q_1(x)-\int h_1(x)q_1(x)dx_1,\ \lambda q_2(x)-\int h_2(x)q_2(x)dx_2\right)=0$$

the solution of the equation (23), where Φ is an arbitrary continuous function of two variables

$$q_i(x)=\exp\left[\int\left[\left(\frac{\partial\gamma_4}{\partial x_i}\right)^{-1}(\Delta\gamma_4-\nu\gamma_4)dx_i\right]\right],$$

$$h_i(x)=\left[\alpha\gamma_4(x)-\omega_2(\eta_4)F_{04}(x)\right]\left(\frac{\partial\gamma_4}{\partial x_i}\right)^{-1},\quad \left(\frac{\partial\gamma_4}{\partial x_i}\right)^{-1}\neq 0,\quad i=1,2.$$

Cite an example on instability of the solution to the problem (1)-(5). Determine $a_{22}(x)$ and $\lambda_2(x)$ in a plane inverse stationary thermo-elasticity problem.

Let a considered domain be the square $0\le x\le 1,\ 0\le y\le 1$ and be given the functions

$$u_2(x,\eta_2)=\frac{A}{m}(1-\cos 2m\pi x),\quad T(x,\eta_3)=\frac{\gamma_0}{a}(\sin ax-x\sin a),$$

where A,G_0,a are arbitrary constants, m is a natural number.

Then above mentioned formulas admit to define

$$a_{22}(x)=\frac{2m\pi}{\mu\sin^2\pi x}\left(\frac{da_{12}}{dx}\sin 2m\pi x+2m\pi a_{12}(x)\cos 2m\pi\, x\right),$$

$$\lambda_2(x)=-\frac{\alpha}{\mu}+\frac{1}{\mu(\sin ax-x\sin a)}\left(a\cos ax-\sin a-\lambda_1 a^2\sin ax\right). \qquad (24)$$

It follows from (24) that for $m\to\infty$, $a\to\infty$ the functions $\lambda_2(x)$ and $a_{22}(x)$ unrestrictedly increase. Consequenly, the solution of the problem (1)-(5) is unstable. Therefore we need a stable algorithm based on the regularization method.

Remark 1. Determination of any three coefficients from $a_{ij}(x),\ b_{ij}(x),\ i,j=\overline{1,3}$ by giving the others is led to the solution of partial differential equations of the first order of type (18).

2. A problem on definition of thermo-elastic characteristics of orthotropic materials when a considered elastic body consists of two parts, one of which thermo-elastic characteristics are known, may be solved by the used method.

3. The cited discussion admits to determine thermo-elastic characteristics of transversal-isotropic and isotropic materials.

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Received September 3, 1997; Revised June 22, 2000.
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