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OPTIMAL DESIGN OF PLASTIC DISKS IN THE PLANE STRESS CONDITION

Abstract

In the paper there is a solution of the problem on optimal from the point of view of weight, the law of distribution of thickness in the uniform rotating circle disk, when the disk is in the plastic stress condition. The necessary conditions of optimality are used in the form of the maximum principle. The initial variation problem is reduced to the non-linear boundary-value problems of the special form. In the case of piece-wise linear surface of the different regimes of yield and possible optimal conditions are discussed.

1. The uniform rotating circle ring disk of variable thickness is considered. It is supposed that the stress state of the disk has the radial symmetry and corresponds to the plane stress.

Then behavior of the disk is described by the equations [3]

$$\begin{cases} \dfrac{d\sigma_r}{dr} = -\dfrac{1}{h}\left[\sigma_r \dfrac{dh}{dr} + \dfrac{h}{r}(\sigma_r - \sigma_\theta) + \rho\omega^2 rh\right] \\ \dfrac{d\sigma_\theta}{dr} = \dfrac{\sigma_r - \sigma_\theta}{r} - \dfrac{\nu}{h}\sigma_r \dfrac{dh}{dr} - \nu\rho\omega^2 r\ , \end{cases} \tag{1}$$

where σ_r, σ_θ are normal stresses, ν is Poisson's coefficient, ω is the angular velocity of rotation, $h(r)$ is thickness, ρ is density of the material.

As the criterion of optimality we take the minimum of weight

$$W(h) = \int_a^b 2\pi \cdot gh(r)dr\ , \tag{2}$$

where a = inner radius of the disk, b = external radius. We take the boundary-conditions

$$\sigma_r = 0 \quad \text{for} \quad r = a \quad \text{and} \quad r = b\ . \tag{3}$$

Assume the material of the disk is subjected to the yield of the form

$$F(\sigma_r, \sigma_\theta) \le \sigma_0^2\ , \tag{4}$$

where σ_0 is a yield point under simple tension.

The restriction exists for the variable thickness of the disk:

$$h(r) \ge h_0\ . \tag{5}$$

As the control function we take $h(r)$. Let's complete Hamilton's function:

$$\begin{aligned} H = 2\pi\rho h(r)\lambda_0 - \frac{\lambda_1}{h}\left[\sigma_r \frac{dh}{dr} + \frac{h}{r}(\sigma_r - \sigma_\theta) + g\omega^2 rh(r)\right] + \\ + \lambda_2\left[\frac{\sigma_r - \sigma_\theta}{r} - \frac{\nu}{h}\sigma_r \frac{dh}{dr} - \nu\rho\omega^2 r\right]. \end{aligned} \tag{6}$$

Since the optimal trajectory is on the bound of the area of restrictions (4), then in the considered case namely the "restricted" maximum principle holds [1-2]. So the adjoined equations are written in the form

$$d\bar{\lambda} = -\bar{\lambda}A + \lambda B\left(\frac{\partial P}{\partial \dot{h}}\right)^{-1} \nabla x P, \tag{7}$$

where $\lambda = (\lambda_0(r), \lambda_1(r), \lambda_2(r))$.

$$A = \frac{\partial f}{\partial x}; \quad B = \frac{\partial f}{\partial \dot{h}}; \; x = (\sigma_r, \tau_\theta, r); \nabla_x P = \left\{\frac{\partial P}{\partial \sigma_r}; \frac{\partial P}{\partial \sigma_\theta}\right\},$$

$$P(x) = \nabla F f = -\frac{1}{h}\left[\sigma_r \frac{dh}{dr} + \frac{h}{r}(\sigma_r - \sigma_\theta) + \rho\omega^2 r h\right]\frac{\partial F}{\partial \sigma_r} +$$

$$+\left[\frac{\sigma_r - \sigma_\theta}{r} - \frac{v}{h}\sigma_r \frac{dh}{dr} - v\rho\omega^2 r\right]\frac{\partial F}{\partial \sigma_\theta}.$$

As the phase velocity of the moving along trajectory of the point at every moment of time is tangential to the bound, so

$$P(\sigma_r, \sigma_\theta, h, h', r) = 0. \tag{8}$$

Solving (8) with respect to $h(r)$ we obtain

$$h(r) = h_0 \exp - \left\{\int \frac{1}{\sigma_r}\left[\frac{\sigma_r - \sigma_\theta}{r}\frac{F_{\sigma_r} - F_{\sigma_\theta}}{F_{\sigma_r} + vF_{\sigma_\theta}} + \rho\omega^2 r\right]dr\right\}, \quad r \in [r_e, r_p], \tag{9}$$

where $F_{\sigma_r}, F_{\sigma_\theta}$ are partial derivatives by σ_r, σ_θ correspondingly.

Substituting (9) in (1) we have

$$\begin{cases} \dfrac{d\sigma_r}{dr} = -\dfrac{1+v}{r}\dfrac{(\sigma_r - \sigma_\theta)F_{\sigma_\theta}}{F_{\sigma_r} + vF_{\sigma_\theta}} \\ \dfrac{d\sigma_\theta}{dr} = -\dfrac{1+v}{r}\dfrac{(\sigma_r - \sigma_\theta)F_{\sigma_r}}{F_{\sigma_r} + vF_{\sigma_\theta}} \end{cases} \tag{10}$$

The joined equations (7) taking into account the accepted denotations are written in the form

$$\begin{cases} \dfrac{d\lambda_1}{dr} = \dfrac{\lambda_1 - \lambda_2}{r} + \dfrac{\lambda_1 + v\lambda_2}{F_{\sigma_r} + vF_{\sigma_\theta}}\left[\dfrac{F_{\sigma_r} - F_{\sigma_\theta}}{r} + \dfrac{d}{dr}\left(F_{\sigma_r}\right)\right] \\ \dfrac{d\lambda_2}{dr} = -\dfrac{\lambda_1 - \lambda_2}{r} + \dfrac{\lambda_1 + v\lambda_2}{F_{\sigma_r} + vF_{\sigma_\theta}}\left[\dfrac{F_{\sigma_r} - F_{\sigma_\theta}}{r} + \dfrac{d}{dr}\left(F_{\sigma_\theta}\right)\right] \end{cases} \tag{11}$$

The obtained results are applied to the general yield surface F.

Let

$$F = \sigma_r^2 - \sigma_r\sigma_\theta + \sigma_\theta^2 \tag{12}$$

Then

$$\begin{cases} \dfrac{\partial F}{\partial \sigma_r} = 2\sigma_r - \sigma_\theta \\ \dfrac{\partial F}{\partial \sigma_\theta} = 2\sigma_\theta - \sigma_r \\ \dfrac{d}{dr}\left(F_{\sigma_r}\right) = 2\dfrac{d\sigma_r}{dr} - \dfrac{d\sigma_\theta}{dr} \\ \dfrac{d}{dr}\left(F_{\sigma_\theta}\right) = 2\dfrac{d\sigma_\theta}{dr} - \dfrac{d\sigma_r}{dr} \end{cases} \tag{13}$$

Substituting (13) in (10) and (11), we obtain

$$\begin{cases} \dfrac{d}{dr}(\sigma_r) = \dfrac{(\sigma_r - \sigma_\theta)(\sigma_r - 2\sigma_\theta)(1+v)}{r[(2-v)\sigma_r - \sigma_\theta(1-2v)]} \\ \dfrac{d}{dr}(\sigma_r) = \dfrac{(\sigma_r - \sigma_\theta)(\sigma_r - 2\sigma_\theta)(1+v)}{r[(2-v)\sigma_r - \sigma_\theta(1-2v)]} \end{cases} \tag{14}$$

The boundary conditions for the joined system have the form:

$$\lambda_2(r)\big|_{r=a} = 0 \qquad \lambda_2(r)\big|_{r=b} = 0$$

$$\begin{gathered} \frac{d\lambda_1}{dr} = \frac{\lambda_1 - \lambda_2}{r} + \frac{3(\lambda_1 + v\lambda_2)(\sigma_r - \sigma_\theta)^2(2-v)}{r[\sigma_r(2-v) - \sigma_\theta(1-2v)]^2} \\ \frac{d\lambda_2}{dr} = -\frac{\lambda_1 - \lambda_2}{2} - \frac{3(\lambda_1 + v\lambda_2)(\sigma_r - \sigma_\theta)}{r[\sigma_r(2-v) - \sigma_\theta(1-2v)]^2} \end{gathered} \tag{15}$$

Thus, we have two systems of differential equations (14), (15) and the system of four boundary conditions which with Gamilton's maximum condition give the necessary conditions to the solution of the formulated problem.

For Tresk's yield condition

$$F(\sigma_r, \sigma_\theta) = \max(|\sigma_2|, |\sigma_\theta|, |\sigma_r - \theta_\theta|) = \sigma_0 . \tag{16}$$

In this case $F_{\sigma_r}, F_{\sigma_\theta}$ are constant, so

$$\frac{d}{dr}\left(F_{\sigma_r}\right) = \frac{d}{dr}\left(F_{\sigma_\theta}\right) = 0, \tag{17}$$

Then from (11) and (17) we obtain

$$\frac{d}{dr}\left(-\lambda_1 F_{\sigma_\theta} + \lambda_2 F_{\sigma_r}\right) = \frac{(1+v)\left(F_{\sigma_\theta} + F_{\sigma_r}\right)\left(-\lambda_1 F_{\sigma_\theta} + \lambda_2 F_{\sigma_r}\right)}{r\left(F_{\sigma_r} + vF_{\sigma_\theta}\right)},$$

whence

$$\left|-\lambda_1 F_{\sigma_q} + \lambda_2 F_{\gamma_r}\right| = cr^\alpha, \tag{18}$$

where $\alpha = (1+v)\dfrac{F_{\sigma_2} + F_{\sigma_\theta}}{F_{\sigma_2} + vF_{\sigma_\theta}}$.

Substitution of (9) and (18) in (6) gives

$$H = -2\pi\rho h(r) + c(1+v)\frac{\sigma_r - \sigma_\theta}{F_{\sigma_r} + vF_{\sigma_\theta}} r^{\alpha-1} sign\left[-\lambda_1 F_{\sigma_\theta} + \lambda_2 F_{\sigma_r}\right]. \tag{19}$$

It's required to determine the optimal controls $h^*(r) \ge h_0$ which maximizes (19).

The possible plastic regimes of the work are the followings

1) $\sigma_r = \sigma_0$; $0 \le \sigma_\theta \le \sigma_0$, (20)

2) $\sigma_\theta = \sigma_0$; $0 \le \sigma_r \le \sigma_0$, (21)

3) $\sigma_r = \sigma_\theta = \sigma_0$. (22)

Let's consider the points along AF (fig.1)

$$\sigma_\theta = \sigma_0; \quad 0 \le \sigma_r \le \sigma_0 ; \qquad F_{\sigma_\theta} = 1; \quad F_{\sigma_r} = 0 . \tag{23}$$

Using (23) from (10) we obtain

$$\frac{d\sigma_r}{dr} = \frac{1+\nu}{\nu}\frac{\sigma_0 - \sigma_r}{r},$$

$$\sigma_r = \sigma_0 - k_1 r^{-\frac{1+\nu}{\nu}} \tag{24}$$

where k_1 - integration constant.

Optimal control is determined by substitution (23) and (24) in (9)

$$h = h_0' \exp\left[\left(\ln\frac{r}{a}\right) - \sigma_0 \int_a^r \frac{dr}{r\left(\sigma_0 - k_1 r^{-\frac{1+\nu}{\nu}}\right)} - \rho\omega^2 \int_a^r \frac{rdr}{\left(\sigma_0 - k_1 r^{-\frac{1+\nu}{\nu}}\right)}\right]. \tag{25}$$

Restrictions for $h(r)$ are satisfied by selection of h_0. Let's consider the points along EF (fig.1)

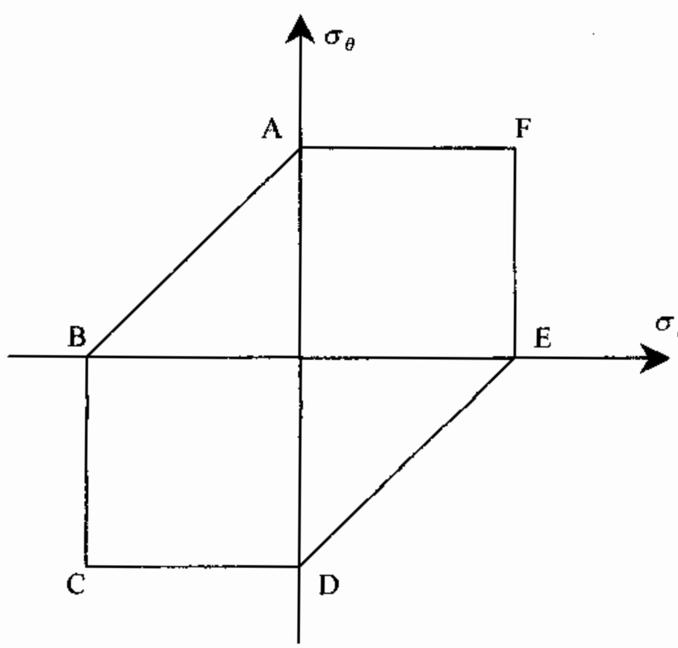

Fig.1

$$\sigma_r = \sigma_0 ; \quad 0 \le \sigma_\theta \le \sigma_0 \qquad F_{\sigma_r} = 1; \quad F_{\sigma_\theta} = 0 . \tag{26}$$

Substitution of (26) in (10) gives

$$\frac{d\sigma_q}{dr} = (1+\nu)\frac{\sigma_0 - \sigma_\theta}{r}\,, \qquad (27)$$
$$\sigma_\theta = \sigma_\theta - k_2 r^{-(1+\nu)}\,,$$

where k_2 - positive integration constant. Optimal control is determined by substitution (26) and (27) in (9)

$$h(r) = h_0^* \exp\left\{\frac{k_2}{(1+\nu)\sigma_0}\left[r^{-(1+\nu)} - a^{-(1+\nu)}\right] - \frac{\rho\omega^2}{2\sigma_0}\left(r^2 - a^2\right)\right\}. \qquad (28)$$

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