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THE IMPROVED CONVERGENCE OF A.A.DORODNITSYN'S METHOD AND THE METHOD OF QUASI-LINEARIZATION FOR NON-LINEAR PROBLEM OF GOURSAT

Abstract

The improved convergence of A.A. Dorodnytsin's method and a quasilinearization method for Goursat's nonlinear problem, when u'_x and u'_y are not contained in a non-linear part is considered in the paper. By means of Dorodnitsyn's method and a quailinearization method the stated boundary- value problem is reduced to the system of ordinary differential equations. Finally, general error for the formulated problem is estimated by a step estimation mechanism.

The differential equation of partial derivatives is met in different branches of science. But to get their solutions in explicit form, in the form of some series or integral is possible only in simple cases. In this connection the main meaning acquires the approximate methods of solutions and their basing. One of these methods is A.A. Dorodnitsyn's method, which was reported in the city of Moscow in 1956 and immediately became wide-applied in different applied problems. It played and continues to play a great role in computation theory. In particular the convergence of Dorodnitsyn's method for linear and quasi-linear problems of Goursat was investigated by us [4] and a number of authors (for example, [3]).

At the present note for getting the order of convergence, when u'_x and u'_y are included in non-linear part the identity in the problem is defined the convergence of is A.A. Dorodnitsyn's successive approximations is investigated by different scheme, which gives us possibility to get the improved order of convergence for non-linear Goursat problem.

Let's consider the following non-linear Goursat problem:

$$u_{xy} = a(x,y)u_x + b(x,y)u_y + F(x,y,u,u'_x,u'_y), \tag{1}$$

$$u(x,0)=0 \ \ (0 \le x \le \mathcal{L}_1), \ \ u(0,y)=0 \ \ (0 \le y \le \mathcal{L}_2), \tag{2}$$

where $a(x,y), b(x,y) (0 \le x \le \mathcal{L}_1, 0 \le y \le \mathcal{L}_2)$ are given continuous functions, $F(x,y,u,u'_x,u'_y)$ is continuous and is continuously differentiable on y,u,u'_x,u'_y: $(x,y,u,u'_x,u'_y) \in D = [0,\mathcal{L}_1]\times[0,\mathcal{L}_2]\times[-\mathcal{L},\mathcal{L}]\times[-\mathcal{L}',\mathcal{L}']$.

Applying the method of A.A.Dorodnitsyn's for the Goursat problem (1)-(2), splitting the rectangle $[0,\mathcal{L}_1]\times[0,\mathcal{L}_2]$ with the chosen $h = \dfrac{\mathcal{L}_2}{N}$ of lines $y = y_n = nh$, $n = 1,\dots,N$ to N strip and interpolating each sub-integral function we get:

$$u'_{n+1}(x) - A^{(0)}_{n+1}(x)u_{n+1}(x) = B^{(0)}_{n+1}(x)u'_n(x) - A^{(0)}_{n+1}(x)u_n(x) + hC^{(0)}_{n+1}(x)\times$$
$$\times\left[F(x,y_{n+1},u_{n+1},u'_{xn+1},u'_{yn+1}) + F(x,y_n,u_n,u'_{xn},u'_{yn})\right] + 2C^{(0)}_{n+1}(x)R_{n+1}(x),$$
$$(n=1,\dots,N), \ \ (0 \le x \le \mathcal{L}_1), \tag{3}$$

$$u_0(x)=0, \ u_{n+1}(0)=0, \ 0 \le x \le \mathcal{L}_1, \ n = 0,1,\dots,N-1, \tag{4}$$

where

$$R_{n+1}(x)=\frac{h}{2}\int_0^1\left[\frac{\partial^3(au)}{\partial x\partial y^2}+\frac{\partial^2(F-a_x u-by)}{\partial y^2}\right]_{y=\tilde{y}_{n+1}}(t-1)tdt,$$

$$y_n<\tilde{y}_{n+1}<\tilde{y}_{n+1},$$

$$un(x)=u(x,y_n),\ \ u_n'(x)=u_x'(x,y_n),$$

$$A_{n+1}^{(0)}(x)=\frac{2b_n(x)}{2-ha_{n+1}(x)},\ B_{n+1}^{(0)}(x)=\frac{2-ha_n(x)}{2-ha_{n+1}(x)},\ C_{n+1}^{(0)}(x)=\frac{1}{2-ha_{n+1}(x)}.$$

Rejecting the remaining term $R_{n+1}(x)$ and applying the quasilinearization method [5] we get:

$$\tilde{u}_{n+1}(x)-A_{n+1}^{(1)}(x)\tilde{u}_{n+1}(x)=B_{n+1}^{(1)}(x)\tilde{u}_n'(x)-A_{n+1}^{(1)}(x)\tilde{u}_n(x)+hC_{n+1}^{(1)}(x)\times$$
$$\times\left[F(x,y_n,\tilde{u}_n,\tilde{u}_{xn}',\tilde{u}_{yn}')+F(x,y_{n+1},\tilde{u}_{n+1},\tilde{u}_{xn+1}',\tilde{u}_{yn+1}')\right],\tag{5}$$

$$\tilde{u}_0(x)=0,\ \tilde{u}_{n+1}(0)=0\ \ (0\le x\le \mathcal{L}_1),\ n=0,1,\dots,N-1,\tag{6}$$

here

$$A_{n+1}^{(1)}(x)=\left(A_{n+1}^{(0)}(x)+hC_{n+1}^{(0)}(x)F_{\tilde{u}}'\right)\Big/\left(1-hC_{n+1}^{(0)}(x)\left(F_{\tilde{u}_x'}'+F_{\tilde{u}_y'}'\right)\right),$$

$$C_{n+1}^{(1)}(x)=C_{n+1}^{(0)}(x)\Big/\left(1-hC_{n+1}^{(0)}(x)\left(F_{\tilde{u}_x'}'+F_{\tilde{u}_y'}'\right)\right),$$

$$B_{n+1}^{(1)}(x)=\left(B_{n+1}^{(0)}(x)-hC_{n+1}^{(0)}(x)F_{\tilde{u}}'\right)\Big/\left(1-hC_{n+1}^{(0)}(x)\left(F_{\tilde{u}_x'}'+F_{\tilde{u}_y'}'\right)\right).$$

Further we'll suppose everywhere, that for the solution of problem (3)-(4) (the existence of solution of this problem is supposed) takes place the estimation:

$$|u_n(x)|\le\mathcal{L},\ |u_n'(x)|\le\mathcal{L}',\ \ 0\le x\le\mathcal{L}_1,\ n=1,2,\dots .$$

I. The estimation for $|\tilde{u}_n(x)|$ and $|\tilde{u}_n'(x)|$.

Theorem. *Let's suppose, that*

$$\alpha=h2C^{(1)}F_0\sum_{k=1}^{n}M_1^k\left(A^{(1)}\right)^{k-1}\left(B^{(1)}+1\right)^{k-1}\left[\sum_{\nu=0}^{n-1}\left(B^{(1)}\right)^{\nu}\right]_{B^{(1)}}^{(k-1)}\mathcal{L}_1^k\Big/(k-1)!k!\le\mathcal{L},$$

$$\alpha'=h2C^{(1)}F_0\left\{\sum_{k=1}^{n}M_1^k\left(A^{(1)}\right)^{k-1}\left(B^{(1)}+1\right)^{k-1}\left(A^{(1)}\left[\sum_{\nu=0}^{n}\left(B^{(1)}\right)^{\nu}\right]_{B^{(1)}}^{(k)}+\right.\right.$$
$$\left.\left.+A^{(1)}\left[\sum_{\nu=0}^{n-1}\left(B^{(1)}\right)^{\nu}\right]_{B^{(1)}}^{(k)}\right)\mathcal{L}^k\Big/k!k!+\sum_{\nu=0}^{n-1}\left(B^{(1)}\right)^{\nu}\right\}\le\mathcal{L}'$$

is fulfilled. Here

$$A^{(1)}=\max_D A_{n+1}^{(1)}(x),\ C^{(1)}=\max_D C_{n+1}^{(1)}(x),\ \ F_0=\max_D\left|F_{n+1}(x)+F_n(x)\right|,$$

$$B^{(1)}=\max_D B_{n+1}^{(1)}(x),\ M_1=\exp A^{(1)}\mathcal{L}_1 .$$

Then it takes place:

$$|\tilde{u}_n(x)|=\alpha\le\mathcal{L},\ \ and\ |\tilde{u}_n'(x)|=\alpha'\le\mathcal{L}',$$

where $\left[\sum_{\nu=0}^{n-1}\left(B^{(1)}\right)^{\nu}\right]_{B^{(1)}}^{(k-1)}$ *means* $(k-1)$ *derivative of the polynomial* $\sum_{\nu=0}^{n-1}\left(B^{(1)}\right)^{\nu}$ *with respect to* $B^{(1)}$.

Proof. Solving the problem (5)-(6) by the successive approximations method when $n=0$ we find:

$$|\widetilde{u}_1(x)| \le M_1 \int_0^x \Big[B^{(1)}|\widetilde{u}_0'| + A^{(1)}|\widetilde{u}_0| + hC^{(1)}\Big(\Big|F\big(x, y_0, \widetilde{u}_0, \widetilde{u}'_{x_0}, \widetilde{u}'_{y_0}\big)\Big| + \Big|F\big(x, y_0, \widetilde{u}_0, \widetilde{u}'_{x_{01}}, \widetilde{u}'_{y_{01}}\big)\Big|\Big)\Big]dc \le$$

$$\le M_1 \int_0^x 2hC^{(1)}F_0 ds\,,$$

$$|\widetilde{u}_{1x}| \le M_1 2hC^{(1)}F_0 x\,,$$

$$|\widetilde{u}_1'(x)| \le A^{(1)}|\widetilde{u}_1(x)| + B^{(1)}|\widetilde{u}_0'(x)| + A^{(1)}|\widetilde{u}_0(x)| + hC^{(1)} \times$$

$$\times \Big(\Big|F\big(x, y_0, \widetilde{u}_0, u'_{x_0}, u'_{y_0}\big)\Big| + \Big|F\big(x, y_0, \widetilde{u}_0, \widetilde{u}'_{x_{01}}, \widetilde{u}'_{y_{01}}\big)\Big|\Big) \le M_1 A^{(1)} 2hC^{(1)}F_0 x + 2hC^{(1)}F_0\,,$$

$$|\widetilde{u}_1'(x)| \le M_1 A^{(1)} 2hC^{(1)}F_0 x + 2hC^{(1)}F_0\,.$$

Now let's suppose, that $|\widetilde{u}_n(x)|$ and $|\widetilde{u}_n'(x)|$ are successively estimated by the following form:

$$|\widetilde{u}_n(x)| \le h2C^{(1)}F_0 \sum_{k=1}^{n} M_1^k \big(A^{(1)}\big)^{k-1}\big(B^{(1)}+1\big)^{k-1}\left[\sum_{\nu=0}^{n-1}\big(B^{(1)}\big)^{\nu}\right]_{B^{(1)}}^{(k-1)} x^k \Big/ (k-1)! \times k!\,, \qquad (7)$$

$$|\widetilde{u}_n'(x)| \le h2C^{(1)}F_0 \left\{\sum_{k=1}^{n} M_1^k \big(A^{(1)}\big)^{k-1}\big(B^{(1)}+1\big)\left(A^{(1)}\left[\sum_{\nu=0}^{n-1}\big(B^{(1)}\big)^{\nu}\right]_{B^{(1)}}^{(k-1)} + \right.\right.$$

$$\left.\left. + A^{(1)}\left[\sum_{\nu=0}^{n-1}\big(B^{(1)}\big)^{\nu}\right]_{B^{(1)}}^{(k)}\right) x^k k! \times k! + \sum_{\nu=0}^{n-1}\big(B^{(1)}\big)^{\nu}\right\} \qquad (8)$$

consequently it follows that

$$|\widetilde{u}_n(x)| \le \mathcal{A}\,,\ |\widetilde{u}_n'(x)| \le \mathcal{A}'\,,\ 0 \le x \le \mathcal{A}_1\,,\ n = 1,2,\ldots\,.$$

With the help of estimations (7) and (8) from the solutions of problems (5)-(6) we have:

$$|\widetilde{u}_{n+1}(x)| \le M_1 B^{(1)} 2hC^{(2)}F_0 \sum_{k=1}^{n} M_1^k \big(A^{(1)}\big)^{k-1}\big(B^{(1)}+1\big)^{k-1}\left[\big(B^{(1)}A^{(1)} + A^{(1)}\big)\sum_{\nu=0}^{n-1}\big(B^{(1)}\big)^{\nu}\right]_{B^{(1)}}^{(k)} \times$$

$$\times x^{k+1} \Big/ k! \times (k+1)! + M_1 B^{(1)} 2hC^{(1)}F_0 \sum_{n=0}^{n-1}\big(B^{(1)}\big)^{\nu} x + M_1 A^{(1)} 2hC^{(1)}F_0 \times$$

$$\times \sum_{k=1}^{n} M_1^k \big(A^{(1)}\big)^{k-1}\big(B^{(1)}+1\big)^{k-1}\left[\sum_{\nu=0}^{n-1}\big(B^{(1)}\big)^{\nu}\right]_{B^{(1)}}^{(k-1)} x^{k+1} \Big/ (k-1)! \times (k+1)! + M_1 2hC^{(1)}F_0\,.$$

Grouping the similar terms and taking into account the identity:

$$B^{(1)}\left[\sum_{\nu=0}^{n-2}\big(B^{(1)}\big)^{\nu}\right]_{B^{(1)}}^{(k)} \Big/ k! + \left[\sum_{\nu=0}^{n-2}\big(B^{(1)}\big)^{\nu}\right]_{B^{(1)}}^{(k-1)} \Big/ (k-1)! = \left[\sum_{\nu=0}^{n-2}\big(B^{(1)}\big)^{\nu}\right]_{B^{(1)}}^{(k)} \Big/ k! \qquad (9)$$

we get

$$|\widetilde{u}_{n+1}(x)| \le 2hC^{(1)}F_0 \sum_{k=1}^{n+1} M_1^k \big(A^{(1)}\big)^{k-1}\big(B^{(1)}+1\big)^{k-1}\left[\sum_{\nu=0}^{n}\big(B^{(1)}\big)^{\nu}\right]_{B^{(1)}}^{(k-1)} x^k \Big/ (k-1)! \times k!\,.$$

Now using the estimation (7), (8) and taking into account the identity (9) from the solutions of problems (5)-(6) we get estimations for $|\widetilde{u}'_{n+1}(x)|$

$$|\widetilde{u}'_{n+1}(x)| \le 2hC^{(1)}F_0 \left\{\sum_{k=1}^{n+1} M_1^k \big(A^{(1)}\big)^{k-1}\big(B^{(1)}+1\big)^{k-1}\left(A^{(1)}\left[\sum_{\nu=1}^{n}\big(B^{(1)}\big)^{\nu}\right]_{B^{(1)}}^{(k)} + \right.\right.$$

$$+A^{(1)}\left[\sum_{\nu=0}^{n}\left(B^{(1)}\right)^{\nu}\right]_{B^{(1)}}^{(k)}\Bigg)x^k \Big/ k!\times k! + \sum_{\nu=0}^{n}\left(B^{(1)}\right)^{\nu}\Bigg\}.$$

Thus, because of the mathematical induction method for all n:

$$\left|\widetilde{u}_n(x)\right| \le h2C^{(1)}F_0\sum_{k=1}^{n}M_1^k\left(A^{(1)}\right)^{k-1}\left(B^{(1)}+1\right)^k\left[\sum_{\nu=1}^{n-1}\left(B^{(1)}\right)^{\nu}\right]_{B^{(1)}}^{(k-1)}x^k \Big/ (k-1)!\times k!,$$

$$\left|\widetilde{u}_n'(x)\right| \le h2C^{(1)}F_0\Bigg\{\sum_{k=1}^{n}M_1^k\left(A^{(1)}\right)^{k-1}\left(B^{(1)}+1\right)^{k-1}\Bigg(\left[\sum_{\nu=0}^{n}\left(B^{(1)}\right)^{\nu}\right]_{B^{(1)}}^{(k)} +$$

$$+\left[\sum_{\nu=0}^{n-1}\left(B^{(1)}\right)^{\nu}\right]_{B^{(1)}}^{(k)}\Bigg)x^k \Big/ k!\times k! + \sum_{\nu=0}^{n-1}\left(B^{(1)}\right)^{\nu}\Bigg\}$$

takes place.

Finally we get that for all n:

$$\left|\widetilde{u}_n(x)\right| \le \mathcal{L},\ \left|\widetilde{u}_n'(x)\right| \le \mathcal{L}',\quad \left(0 \le x \le \mathcal{L}_1\right)$$

takes place consequently the theorem is proved completely.

II. The estimation for $\left|\widetilde{u}_{n+1}(x)-\widetilde{u}_n(x)\right|$ and $\left|\widetilde{u}_{n+1}'(x)-\widetilde{u}_n'(x)\right|$. Let's introduce the notation:

$$\delta_{n+1}(x) = \widetilde{u}_{n+1}(x) - \widetilde{u}_n(x),\quad n = 0,1,\dots.$$

From the problems (5)-(6) we get the boundary value problem for $\delta_n(x)$, applying analogical scheme as in I to this received boundary value problem we estimate $\delta_n(x)$ and $\delta_n'(x)$ by the next form:

$$\left|\delta_n(x)\right| \le kh^2\sum_{k=1}^{n}M_2\left(B^{(2)}A^{(2)}+C^{(2)}\right)^{k-1}\left[\sum_{\nu=0}^{n-1}\left(B^{(2)}\right)^{\nu}\right]_{B^{(2)}}^{(k-1)}\mathcal{L}_1^k \Big/ (k-1)!\times k!,$$

$$\left|\delta_n'(x)\right| \le kh^2\Bigg\{\sum_{k=1}^{n}M_2^k\left(B^{(2)}A^{(2)}+C^{(2)}\right)^{k-1}\Bigg(A^{(2)}\left[\sum_{\nu=0}^{n}\left(B^{(2)}\right)^{\nu}\right]_{B^{(2)}}^{(k)} +$$

$$+C^{(2)}\left[\sum_{\nu=0}^{n-1}\left(B^{(2)}\right)^{\nu}\right]_{B^{(2)}}^{(k)}\mathcal{L}_1^k \Big/ k!\times k! + \sum_{\nu=0}^{n-1}\left(B^{(2)}\right)^{\nu}\Bigg\}$$

hence for $B^{(2)} < 1$ we have

$$\left|\delta_n(x)\right| \le k_0^* h^2,\quad \left|\delta_n'(x)\right| \le k_1^* h^2,\quad n = 0,1,\dots,\quad 0 \le x \le \mathcal{L}_1$$

k_0^* and k_1^* depends on numbers $M_2, B^{(2)}, A^{(2)}, K, C^{(2)}$.

III. The general estimation of errors of the methods. Let's introduce the notation

$$\gamma_n(x) = u_n(x) - \widetilde{u}_n(x)$$

and substracting from the problems (3)-(4) the (4)-(5) we get the boundary value problem for the error $\gamma_n(x)$ in the following form:

$$\gamma_{n+1}'(x) - A_n^{(0)}(x)\gamma_{n+1}(x) = B_{n+1}^{(0)}(x)\gamma_n'(x) - A_{n+1}^{(0)}\gamma_n(x) + hC_{n+1}^{(0)}(x)\left[F_{\widetilde{u}}'(x, y_n, \widetilde{u}_n + \theta\gamma_n, \widetilde{u}_{xn}' +\right.$$

$$+\theta\gamma'_n,\widetilde{u}'_{yn}+\theta\gamma'_n\big)\gamma_n+F'_{\widetilde{u}'}\gamma'_n(x)\big]+hC_{n+1}\big[F'_{\widetilde{u}}\big(x,y_{n+1},\widetilde{u}_{n+1}+\theta\gamma_{n+1},\widetilde{u}'_{xn+1}+\theta\gamma'_{n+1},\widetilde{u}_{yn+1}+\theta\gamma_{n+1}\big)\times$$

$$\times\gamma'_{n+1}(x)+F'_{\widetilde{u}'}\gamma'_{n+1}(x)\big]+hC^{(0)}_{n+1}(x)\left[\frac{\left(\widetilde{u}'_{n+1}(x)-\widetilde{u}'_n(x)\right)^2}{2}F''_{\widetilde{u}\widetilde{u}}\left(x,y_{n+1},\xi_n,\eta_n,\psi_n\right)+\right.$$

$$+\left(\widetilde{u}_{n+1}(x)-\widetilde{u}_n(x)\right)\left(\widetilde{u}'_{n+1}(x)-\widetilde{u}'_n(x)\right)F''_{\widetilde{u}\widetilde{u}'}\left(x,y_{n+1},\xi_n,\eta_n,\psi_n\right)+\frac{\left(\widetilde{u}'_{n+1}(x)-\widetilde{u}'_n(x)\right)}{2}\times$$

$$\times F''_{\widetilde{u}'\widetilde{u}'}\left(x,y_{n+1},\xi_n,\eta_n,\psi_n\right)\Big]+2C^{(0)}_{n+1}(x)R_{n+1}(x),$$

$$\gamma_0(x)=0\,,\ \gamma_{n+1}(0)=0\,,\ \left(0\le x\le \mathcal{L}_1\right),\ \ n=0,1,\dots,N-1\,.$$

Solving this problem by the analogical scheme, as in I we'll get general estimations of errors of the methods in the following form:

$$|\gamma_n(x)|\le\sum_{k=1}^{n}M_3^k\left(B^{(3)}A^{(3)}+C^{(3)}\right)^{k-1}\left(\left[\sum_{\nu=0}^{n-1}\left(B^{(3)}\right)^{\nu}\left(\delta_{n-\nu}+\delta'_{n-\nu}\right)\right]_{B^{(3)}}^{(k-1)}+\right.$$

$$\left.+R^*\left[\sum_{\nu=1}^{n-1}\left(B^{(3)}\right)^{\nu}\right]_{B^{(3)}}^{(k-1)}\right)\mathcal{L}_1^k\Big/(k-1)!\times k!,$$

$$|\gamma'_n(x)|\le\sum_{k=1}^{n}M_3^k\left(B^{(3)}A^{(3)}+C^{(3)}\right)^{k-1}\left(A^{(3)}\left[\sum_{\nu=1}^{n}\left(B^{(3)}\right)^{\nu}\delta_{n-(\nu-1)}\right]_{B^{(3)}}^{(k)}+\right.$$

$$\left.+C^3\left[\sum_{\nu=0}^{n-1}\left(B^{(3)}\right)^{\nu}\delta_{n-(\nu-0)}\right]_{B^{(3)}}^{(k)}\right)\mathcal{L}_1\Big/k!\times k!+\sum_{\nu=0}^{n-1}\left(B^{(3)}\right)\delta_{n-\nu}\,.$$

So, we have

$$|\gamma_n(x)|\le k_0^{**}h^3\,,\ n=0,1,\dots,\ \ 0\le x\le\mathcal{L}_1\,,$$

where k_0^{**} depends on numbers $M_3,B^3,A^{(3)},C^{(3)}$.

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